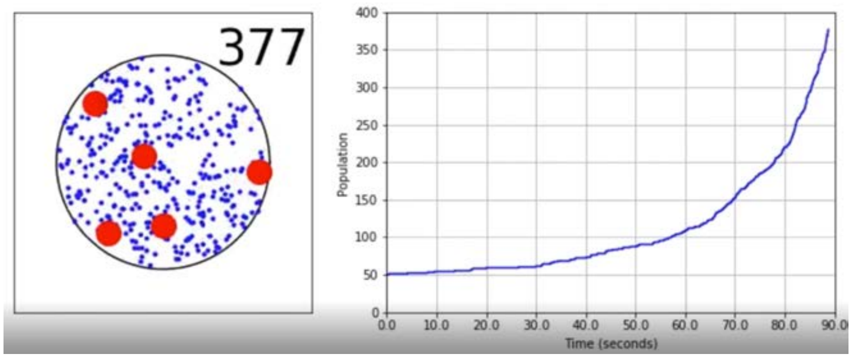
The collision model

Start by viewing the animation. Roughly speaking, when two particles "collide" they produce an offspring particle, which we signal with a red flash. The offspring goes off in a random direction and becomes a new adult.



The question is: is this exponential growth? That is, does the population grow at a rate proportional to its size N? That's what we had in the lightning model. Is that the case here?

When we show this to a grade 11 class and ask if it is exponential growth they talk about it warily for a while but ultimately vote YES (but with many abstentions). Well it grows the way most biological populations grow, so doesn't that make it exponential?

When I ask someone to "convince me" I get an explanation like this: if I am a random individual, the probability that I will have a collision in any small unit of time is proportional to the number of individuals (or maybe to the number of *other* individuals) in the population so doesn't that mean growth rate proportional to size? So it's exponential?

All true! Very good! But that's *individual* growth––growth rate *per individual*. Since there are N individuals, *population* growth rate will be N times this. So population growth rate is really proportional to something like N^2. And that's not exponential.

To proceed with the analysis we need a more detailed description of the process that generates the curve. First of all, as always, particles are moving points. But they are not moving continuously. What we are viewing is "stop-action animation." You see a new screen every 1/25th of a second. At each screen, every particle has a position and a direction of movement. First we look at all the positions and take note of all the cases in which a pair of particles are less than $\delta = 0.02$ apart. For every such case we create a new particle with a new random direction and show a red flash. Then secondly, we move every particle, following its direction of movement the distance it is supposed to move in 1/25th of a second. Since particle speed is 4 cm/s, that's a distance of 0.16 cm. That gives us the next screen.

This is a very interesting problem. It's a great example for a grade 11 student in the process of understanding exponential growth (because it isn't exponential growth!) And then, later on, it's a fine problem for a university calculus student to solve.

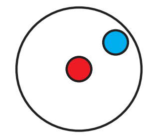
"The exception proves the rule." To fully understand a rule it is important to meet cases where it doesn't apply.

For this animation the parameters are:

radius of the circle: $r = 5$
Particle speed: $v = 4$
collision proximity: $\delta = 0.02$
frame freq: 25 frames/s
initial pop $N = 50$

The dynamic equation.

We will follow the same kind of analysis we used for the lightning model. Consider a focal (red) particle. At any frame (screen), what will be the expected number ε of "collisions" that it is involved in? A collision will happen if a random (blue) particle finds itself inside a circle of radius 0.02 about the focal particle. Now there are N–1 other (blue) particles, all independent of one another so the expected number of these that are inside that circle will be



$$\varepsilon = (N-1)\frac{\pi\delta^2}{\pi r^2} = (N-1)\frac{\pi(0.02)^2}{\pi(5)^2} = (N-1)\frac{0.0004}{25} = \frac{N-1}{62500}$$

This applies for every (red) particle in the population and there are N of these. Then the change in population size is

$$\Delta N = \frac{\varepsilon N}{2} = \frac{N}{2}\left(\frac{N-1}{62500}\right) = \frac{N(N-1)}{125000}$$

Note the 2 in the denominator. That's because we are double counting. That collision will get counted as a reproduction for both the red and the blue particle but only the one offspring will result.

That division by 2 requires careful thinking, but it happens a lot in this kind of argument. In working this problem with different groups of students, I have been surprised (and pleased) by how often this issue is raised by the students in the course of their analysis.

Note: be careful with the units of time. ΔN is the expected change in population size every time we make a count of the collisions and that's every 1/25th of a second.

Is this exponential growth?––certainly not. Population growth over each time step is not proportional to N. It's proportional to N(N–1).

This might well be as far as we want to go with grade 11 students. But I will carry on with further analysis that might be of interest to some students in grade 11 or grade 12, and is squarely in the curriculum of a first-year calculus student in university.

An equation for N

Can we use this equation to get a formula for N at any time? Let's start by being careful about the units of time. The unit of time for the change in size above is 1/25th of a second. I'm going to use the variable τ ("tau" is the Greek letter for t) for time measured in 1/25th of a second. Then I can reserve t for time in seconds.

To write the ΔN equation more explicitly, I will use N_τ for the population size at any time τ (in 1/25th seconds). Then the growth equation

$$\Delta N = \frac{N(N-1)}{125000}$$

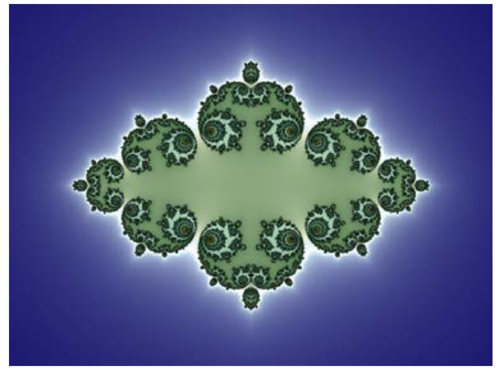
can be written as a recursive equation:

$$N_{\tau+1} = N_\tau + \frac{N_\tau(N_\tau - 1)}{125000} \qquad N_0 = 50$$

By adding the initial condition, I get a formulation that completely specifies the population size at any time τ (τ a positive integer).

Take note that this recursive equation updates N every 1/25th of a second. Thus, following this equation, the population size after one second would be N_{25}, after two seconds N_{50}, etc.

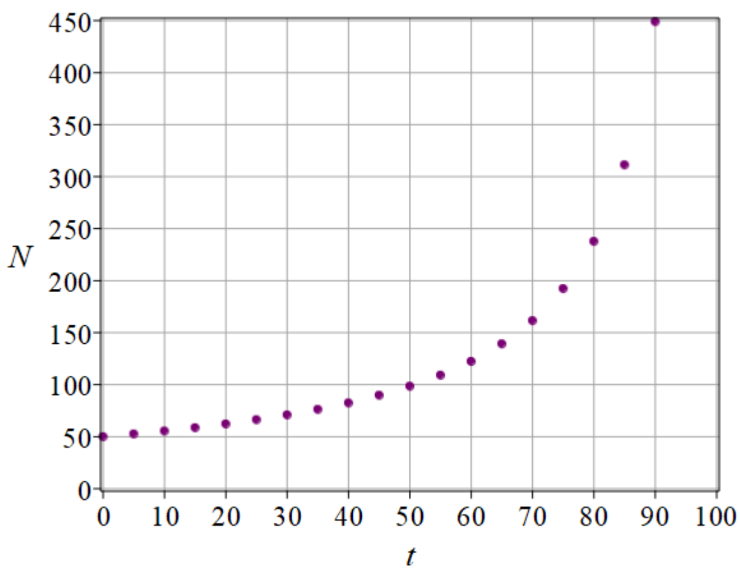
Can we "solve" this equation? That is, can we find a formula for N at any time τ? In fact the answer is no––it can't be solved. It turns out that recursive equations can almost never be solved. In sections 10 and 11 in which we look at interest rates and annuities, the recursive equations that arise are linear and these *can* be solved. But otherwise only exceptional cases can be handled. This one is quadratic and there is no general analysis available. However, even though they can't be solved analytically, quadratic recursive equation can be numerically tracked to give us beautiful images such as the Mandelbrot set and its many variations.



What *can* we do? Well we can certainly track the change in size numerically, with a program or a spreadsheet. That's an important procedure and I want every student to get the chance to do it.

I used Excel to track the population size for 200 seconds. That required 25×200 = 5000 iterations and the results are tabulated at the right and plotted below. Note that the time scale measures time in seconds.

Time t	Size N
0	50
5	52.6
10	55.4
15	58.6
20	62.2
25	66.3
30	70.9
35	76.2
40	82.4
45	89.8
50	98.5
55	109.1
60	122.3
65	139.2
70	161.5
75	192.3
80	237.7
85	311.3
90	450.9



That might look at first glance like an exponential curve, but it is not. Ask the class if they can suggest a simple "eyeball" check to show that it isn't exponential.

Here's one.––look at the doubling time. It takes 50 seconds to go from 50 to 100. Another 25 seconds brings it to 200. And less than 15 more seconds gives us 400. The doubling time is certainly not constant. In fact each doubling step seems to take about half as long as the previous step.

There are different kinds of data in this section. The data in the table above, and plotted at the left, are from our discrete-time theoretical model.

$$N_{\tau+1} = N_\tau + \frac{N_\tau(N_\tau - 1)}{125000}$$

Using a continuous approximation.
One interesting thing we *can* do is change from discrete time to continuous time and replace what is essentially a *difference* equation with a *differential* equation.

$$\frac{\mathrm{d}N}{\mathrm{d}\tau} = \frac{N(N-1)}{125000}$$

Can we solve this equation?

Yes we can. In fact because it has a special form called "separable," all we need is integration.

Put the N's on one side and τ on the other.

$$\frac{\mathrm{d}N}{N(N-1)} = \frac{\mathrm{d}\tau}{125000}$$

Now we integrate both sides. First we rewrite the LHS:

$$\left(\frac{1}{N-1} - \frac{1}{N}\right)\mathrm{d}N = \frac{\mathrm{d}\tau}{125000}$$

$$\ln(N-1) - \ln(N) = \frac{\tau}{125000} + c$$

$$\ln\left(\frac{N-1}{N}\right) = \frac{\tau}{125000} + c$$

Exponentiate

$$\frac{N-1}{N} = e^c \cdot e^{\frac{\tau}{125000}} = ke^{\frac{\tau}{125000}}$$

Put in the initial condition $\tau = 0$, $N = 50$:

$$\frac{49}{50} = k$$

$$\frac{N-1}{N} = \frac{49}{50}e^{\frac{\tau}{125000}}$$

Solve for N

$$50(N-1) = 49Ne^{\frac{\tau}{125000}}$$

$$\left(50 - 49e^{\frac{\tau}{125000}}\right)N = 50$$

$$N = \frac{50}{50 - 49e^{\frac{\tau}{125000}}} = \frac{1}{1 - 0.98e^{\frac{\tau}{125000}}}$$

Switch to time t in seconds: $\tau = 25t$

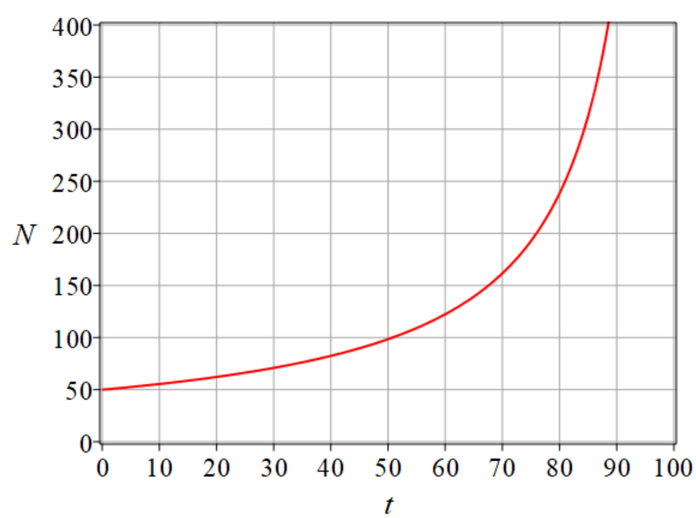
$$N = \frac{1}{1 - 0.98e^{\frac{25t}{125000}}} = \frac{1}{1 - 0.98e^{\left(\frac{t}{5000}\right)}}$$

The graph is plotted at the right.

Now we are in the domain of first-year university calculus. But nevertheless, some grade 11 students would be interested to see this.

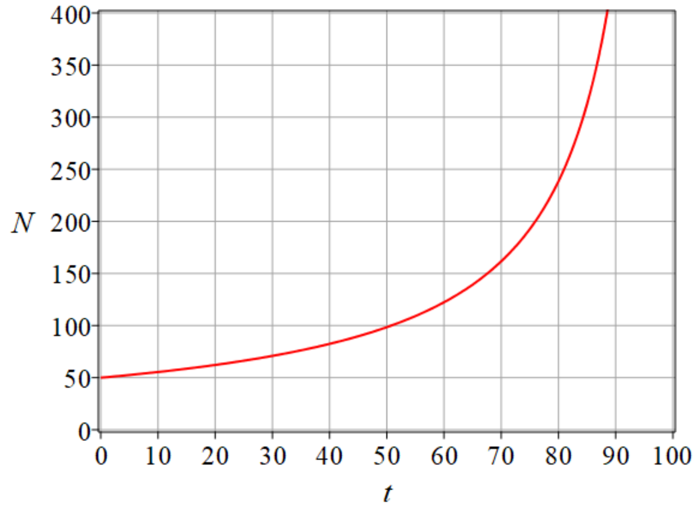
Whenthe time interval is short, an equation for discrete growth can be approximated by an equation for continuous growth, that is by a *differential* equation. And the power of calculus allows us to solve a large variety of differential equations.

The data in the red curve below are plotted from our continuous-time theoretical model. This is an approximation of the discrete-time model above.



Around $t = 70$, the curve starts to grow very fast. In fact it has a vertical asymptote when

$$0.98e^{\left(\frac{t}{5000}\right)} = 1$$
$$t \approx 101$$



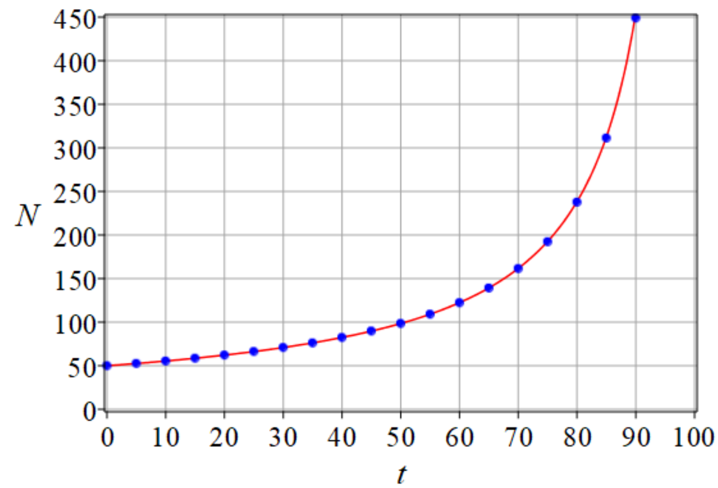
Finally we put the two graphs, discrete

$$N_{\tau+1} = N_\tau + \frac{N_\tau(N_\tau - 1)}{125000}$$

and continuous

$$N = \frac{1}{1 - 0.98e^{\frac{\tau}{125000}}}$$

together on the same set of axes. [But note that the time axis is t whereas the formulae use τ.]



The fit is surprisingly good. But there's one interesting surprise. As I said above, the red curve is headed for a vertical asymptote at $t =$ 101. But the blue dots most certainly are not, that is, τ can march on forever. [Take a minute to think about this. It's an interesting distinction, and one that is of mathematical interest.]

Here we have the data from the two models, discrete and continuous, plotted on the same set of axes. Finally below, the ragged blue curve is empirical data from a run of the animation.

Finally the graph at the bottom plots the theoretical curve (red) on top of data (blue) taken from a run of the animation. In fact if we run the animation, we will get a wide variety of curves some differing greatly from the one shown. Some runs get past 400 before $t = 70$, while others have grown only to $N = 200$ at $t = 90$.

For the graph below, we simply chose a run that gave a good fit to the theoretical model. It was not hard to find such a run.