

Proof-Writing Activities

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First-Year Math and Stats in Canada Online Meetup: Part 2 of How to teach
'em proofs in first-year math and beyond? May 2021

A Resource Bank for Writing Intensive Mathematics Courses



Flawed Proofs:

<http://proof.ucalgaryblogs.ca/>

- Research has shown that an effective way for students to improve their mathematical writing is to read, analyze, and correct proofs that contain (serious) errors.
- Jerrod Smith, Kimberly Golubeva, Christian Bagshaw, and I created a resource bank containing flawed proofs (inspired by former students' proofs).
- The resource also contains error classifications, 'corrected' proofs, and sample activities, assessments, and rubrics that can be used in conjunction with the flawed proofs.
- Topics include linear algebra, discrete math, mathematical induction, elementary topics (e.g., modular arithmetic), and several advanced topics (e.g., real analysis, ring theory).

- Consider the following prompt and flawed proof. How would you describe the errors?

Exercise 5.5.1

Let C be a nonzero $m \times n$ matrix and let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_\ell$ be nonzero vectors in \mathbb{R}^n . Prove that if $\{C\mathbf{v}_1, C\mathbf{v}_2, \dots, C\mathbf{v}_\ell\}$ is linearly independent, then $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_\ell\}$ is linearly independent.

Flawed Proof 5.5.1

Suppose $\{C\mathbf{v}_1, C\mathbf{v}_2, \dots, C\mathbf{v}_\ell\}$ is linearly independent. Then

$$a_1(C\mathbf{v}_1) + a_2(C\mathbf{v}_2) + \dots + a_\ell(C\mathbf{v}_\ell) = \vec{0}_m$$

implies that $a_1 = a_2 = \dots = a_\ell = 0$. Now, we have that

$$\vec{0}_m = a_1(C\mathbf{v}_1) + a_2(C\mathbf{v}_2) + \dots + a_\ell(C\mathbf{v}_\ell) = C(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_\ell\mathbf{v}_\ell)$$

and, since $C \neq \mathbf{0}$, it follows that $\vec{0}_n = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_\ell\mathbf{v}_\ell$. We know that $a_1 = a_2 = \dots = a_\ell = 0$. This implies that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_\ell\}$ is linearly independent. \square

- We classified errors using Strickland & Rand's Proof-Error Coding Scheme.*

Error codes

- Fundamental Logical Order (F-Log)
- Novice Local Omission (N-O)
- Content False Implication (C-FI)

5.5.1 Error classification

There are several errors in the Flawed Proof 5.5.1.

F-Log: The Flawed Proof 5.5.1 incorrectly begins by considering a linear combination of the vectors $C\mathbf{v}_1, C\mathbf{v}_2, \dots, C\mathbf{v}_\ell$. In order to prove that the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_\ell\}$ is linearly independent, we must prove that: if $\vec{0}_n = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_\ell\mathbf{v}_\ell$ for some scalars $a_i \in \mathbb{R}$, $1 \leq i \leq \ell$, then $a_i = 0$ for all $1 \leq i \leq \ell$. In order to prove this statement, we must begin with the assumption that “ $\vec{0}_n = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_\ell\mathbf{v}_\ell$ for some scalars $a_i \in \mathbb{R}$, $1 \leq i \leq \ell$ ”.

N-O: The coefficients a_1, a_2, \dots, a_ℓ are undefined.

C-FI: The implication “since $C \neq \mathbf{0}$, it follows that $\vec{0}_n = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_\ell\mathbf{v}_\ell$ ” is false. In this setting, the claim that $\vec{0}_n = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_\ell\mathbf{v}_\ell$ is equivalent to stating that $\text{null}(C) = \{\vec{0}_n\}$. But the fact that C is a nonzero matrix does not imply that the nullspace of C is equal to $\{\vec{0}_n\}$. For example, the nonzero matrix $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has nullspace $\text{null}(C) = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

C-FI: The implication “We know that $a_1 = a_2 = \dots = a_\ell = 0$. This implies that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_\ell\}$ is linearly independent.” is false. The fact that $\vec{0}_n = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \dots + 0\mathbf{v}_\ell$ does not imply that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_\ell\}$ is linearly independent. To prove that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_\ell\}$ is linearly independent, one must show that $a_1 = a_2 = \dots = a_\ell = 0$ is the **only solution** to the equation $\vec{0}_n = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_\ell\mathbf{v}_\ell$, where $a_i \in \mathbb{R}$, $1 \leq i \leq \ell$.

*S. Strickland and B. Rand, A framework for identifying and classifying undergraduate student proof errors, PRIMUS 26 (2016), no. 10, 905-921.

5.5.2 Corrected proof

The following is a corrected version of Flawed Proof 5.5.1.

Proof 5.5.1

Let C be a nonzero $m \times n$ matrix and let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_\ell$ be nonzero vectors in \mathbb{R}^n . Suppose that $\{C\mathbf{v}_1, C\mathbf{v}_2, \dots, C\mathbf{v}_\ell\}$ is linearly independent. Suppose that

$$\vec{0}_n = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_\ell\mathbf{v}_\ell$$

for some scalars $a_i \in \mathbb{R}$, $1 \leq i \leq \ell$. Multiplying this equation by the $m \times n$ matrix C we obtain

$$\begin{aligned}\vec{0}_m &= C\vec{0}_n \\ &= C(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_\ell\mathbf{v}_\ell) \\ &= a_1(C\mathbf{v}_1) + a_2(C\mathbf{v}_2) + \dots + a_\ell(C\mathbf{v}_\ell).\end{aligned}$$

Since $\{C\mathbf{v}_1, C\mathbf{v}_2, \dots, C\mathbf{v}_\ell\}$ is linearly independent, it must be the case that $a_i = 0$ for all $1 \leq i \leq \ell$. Thus, the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_\ell\}$ is linearly independent. \square

Second-Year Linear Algebra



Context:

- Second-year linear algebra course (~100 students)
- The course serves as an intro to proofs for many students (...some have taken a proofs course and some have not)
- Content: Vector spaces, spanning, linear independence, bases, isomorphisms, orthogonality, change of basis.
- W2021: Students learned the core content asynchronously. We met on Zoom on Wednesdays for a “Q&A Session” and on Fridays for a “Proof-Writing Session” where we discussed flawed proofs.

Example 1:

Suppose that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set in \mathbb{R}^3 . Prove that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

- Take a few minutes to try to prove this.

Example 1:

Suppose that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set in \mathbb{R}^3 . Prove that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

Proof:

Suppose $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set. Suppose that $t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3 = \vec{0}$ for some $t_1, t_2, t_3 \in \mathbb{R}$. By the definition of orthogonality, we know

$$\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1 \cdot \vec{v}_3 = \vec{v}_2 \cdot \vec{v}_3 = 0$$

and

$$\vec{v}_1, \vec{v}_2, \vec{v}_3 \neq \vec{0}.$$

Since these vectors are nonzero, we know $\|\vec{v}_i\|^2 \neq 0$ for all $1 \leq i \leq 3$. Hence, it follows that

$$0 = \vec{0} \cdot \vec{v}_1 = (t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3) \cdot \vec{v}_1 = t_1\vec{v}_1 \cdot \vec{v}_1 = t_1\|\vec{v}_1\|^2 \implies t_1 = 0,$$

$$0 = \vec{0} \cdot \vec{v}_2 = (t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3) \cdot \vec{v}_2 = t_2\vec{v}_2 \cdot \vec{v}_2 = t_2\|\vec{v}_2\|^2 \implies t_2 = 0,$$

$$0 = \vec{0} \cdot \vec{v}_3 = (t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3) \cdot \vec{v}_3 = t_3\vec{v}_3 \cdot \vec{v}_3 = t_3\|\vec{v}_3\|^2 \implies t_3 = 0.$$

Therefore, we have $t_1 = t_2 = t_3 = 0$, and so we can conclude that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent. \square

Example 1:

Suppose that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set in \mathbb{R}^3 . Prove that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

Faulty Proof:

Suppose that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set. Since this set is orthogonal, then for any vectors in the set, their dot product is equal to zero. Now suppose that $t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3 = \vec{0}$. We want to prove that $t_1 = t_2 = t_3 = 0$. We will do this by taking dot products. So we have

$$(t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3) \cdot \vec{v}_1 = t_1\vec{v}_1 \cdot \vec{v}_1 \implies t_1 = 0 ,$$

$$(t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3) \cdot \vec{v}_2 = t_2\vec{v}_2 \cdot \vec{v}_2 \implies t_2 = 0 ,$$

$$(t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3) \cdot \vec{v}_3 = t_3\vec{v}_3 \cdot \vec{v}_3 \implies t_3 = 0 .$$

Thus, it must be the case that $t_1 = t_2 = t_3 = 0$ and so we can conclude that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

Example 1:

Suppose that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set in \mathbb{R}^3 . Prove that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

Faulty Proof:

Suppose that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set. Since this set is orthogonal, then for any vectors in the set, their dot product is equal to zero. Now suppose that $t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3 = \vec{0}$. We want to prove that $t_1 = t_2 = t_3 = 0$. We will do this by taking dot products. So we have

$$\begin{aligned}(t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3) \cdot \vec{v}_1 &= t_1\vec{v}_1 \cdot \vec{v}_1 \implies t_1 = 0, \\(t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3) \cdot \vec{v}_2 &= t_2\vec{v}_2 \cdot \vec{v}_2 \implies t_2 = 0, \\(t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3) \cdot \vec{v}_3 &= t_3\vec{v}_3 \cdot \vec{v}_3 \implies t_3 = 0.\end{aligned}$$

Thus, it must be the case that $t_1 = t_2 = t_3 = 0$ and so we can conclude that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

Errors:

- **Content Vocabulary & Grammar:** Not stating the definition of an orthogonal set correctly when they say, “for any vectors in the set, their dot product is equal to zero.”
- **Novice Vocabulary & Grammar:** They have seemingly dotted both sides of the equation with \vec{v}_i , but didn't include $\vec{0} \cdot \vec{v}_i$ in their computations.
- **Assertion:** The implications “ $\implies t_i = 0$ ” for $1 \leq i \leq 3$ require justification. This holds because we know that the t_i 's are nonzero by definition of an orthogonal set.

Error Classification:

- I used a simplified error-classification with my class. On tests, we'd use this language when grading.

Mathematical Errors:

- False Implication
- False Statement (a statement that's false, but it's not an implication)
- Content Vocabulary & Grammar (misuse of definition or vocabulary)
- Assertion (result stated without proper justification)
- Omitted Section (e.g., a case is missing)
- Misusing a theorem (e.g., applying the converse of a theorem)
- Wrong Method (e.g., if the prompt says to use a certain method and they don't)
- Wrong Problem (e.g., told to prove the converse, but prove something else)

Writing Clarity Errors:

- Notation
 - Rhetorical Vocabulary & Grammar (prose is poorly written)
 - Extraneous Detail (attempted or proved irrelevant results)
- I found that this language helped to clarify expectations (i.e., that we care about writing quality) and made grading easier.

Proof-Writing Reflection:

- Submit a proof attempt to the Discussion Board. (Five throughout semester.)
- Provide constructive feedback to two of your peers proofs using the error-classification language we discussed in class.
- At the end of the semester, submit a critical reflection where you reflect on how your proof-writing skills developed throughout the semester.

Student Observations:

Students reported in their final reflection that:

- reading their peers proofs inspired them to make changes to their own proof-writing (e.g., being more clear about assumptions, making their proofs neater and more organized).
- the activity made their proof-writing on tests better.
- reading their peers proofs and comments was helpful because they saw multiple ways to prove the same statement and made note of errors so that they didn't make the same mistakes.
- they felt very good and pleased with themselves when receiving positive feedback from peers.
- the activity pushed them to spend more time reviewing the course content, because they wanted the feedback from their peers to be good.

First-Year Calculus



Context:

- First-year calculus course (~2000 students in the Fall)
 - Tests consist of a multiple-choice portion and a written portion (graded on both mathematical correctness and clarity of writing).
1. A cylindrical metal can without a top is made to contain $100 \times b \text{ cm}^3$ of liquid. Use the *Global Interval Method* to find the dimensions (radius and height) that will minimize the amount of metal needed to make the can. Leave your final answer as an exact value or round to two decimal places. Include units in your final answer.

Hint: If a cylinder without a top has radius r and height h , then the volume of the cylinder is $\pi r^2 h$ and the surface area is $\pi r^2 + 2\pi r h$.

The *Global Interval Method* is a theorem, and so to receive full points you must clearly explain why the assumptions of the theorem hold, and then clearly explain what the theorem tells us. Explain your solution using full sentences. See *Written Portion Rubric* for more information.

Communication Score: 3pts

Attribute	Criteria	Excellent (3pts)	Very Good (2pts)	Good (1pt)	Needs Improvement (0pts)
Communication Score	Submission is well articulated using proper notation (e.g. all variables are defined, limit notation is used when calculating limits, etc.), prose are used to help the reader navigate (e.g. solution is organized, uses full sentences, computations are introduced, results are clearly articulated), and there is no extraneous detail (e.g. no irrelevant results or computations). The Communication Score may not exceed the Mathematics Score.	The submission is clearly communicated and follows all of the communication criteria (notation, prose, extraneous detail).	The submission is clearly communicated and follows most of the communication criteria (notation, prose, extraneous detail).	The submission is unclearly communicated and follows some of the communication criteria (notation, prose, extraneous detail).	The submission is unclearly communicated and follows few of the communication criteria (notation, prose, extraneous detail).

Mathematics Score: 4 pts

Attribute	Criteria	Excellent (4pts)	Very Good (3pts)	Good (2pt)	Needs Improvement (0-1pts)
Mathematics Score	Submission is mathematically correct (e.g. no false statements, calculations are accurate, understanding of theory is evident), contains an appropriate amount of detail (e.g. assumptions of theorems are checked, claims are justified), and solves the problem .	Submission follows all of the mathematics criteria (correct, appropriate amount of detail, solves the problem).	Submission follows most of the mathematics criteria (correct, appropriate amount of detail, solves the problem).	Submission follows some of the mathematics criteria (correct, appropriate amount of detail, solves the problem).	Submission follows few of the mathematics criteria (correct, appropriate amount of detail, solves the problem).

Written Portion Checklist

To make sure your solution is complete, it may help to ask yourself the following questions:

Communication Score

Notation

- Are all variables defined? (e.g. did you say what x represents)?
- Does your notation follow conventions? (e.g. limit notation, equal signs are included)

Prose

- Is my solution written in full sentences and organized in paragraphs (or does it look like a string of computations)?
- Are my computations introduced so that the reader knows what I am about to do?
- Do I have a concluding sentence so that it's clear what my result is?

Extraneous Detail

- Have I included details that are irrelevant and do not contribute to my result? (e.g. scrap work that isn't necessary to the problem)
- Is my explanation concise, or can I communicate this in an easier way?

Mathematics Score

Correctness

- Have I double-checked my work to make sure my calculations are correct, and I haven't made a mistake?
- Have I rounded a number early in the computation, which could lead to an inaccurate answer?
- Is the terminology I am using accurate?
- If I used a theorem, am I using it correctly?

Appropriate Amount of Detail

- If a fellow classmate who knows a little less than me was reading my solution, would they understand all steps, or should I provide more details?
- If I used a theorem, am I explaining how all assumptions are satisfied?

Solves the Problem

- Have I reread the prompt to make sure that I read it correctly and didn't miss any parts if it is a multipart question?
- Does my concluding sentence completely solve the problem?
- Have I double-checked to make sure that my single PDF contains all of my work, and I haven't accidentally missed a page?

- We break down the “Mathematical Score” for the TAs for each question. The “Communication Score” is holistic.

Math Score (4pts):

Set Up (1pt)	Computation (1pt)	Global Interval Method (1pt)	Final Answer (1pt)
<ul style="list-style-type: none"> • Solving for h in terms of r. (0.5pts) • Putting A in terms of r. (0.25pts) • Understanding that they need to minimize A. (0.25pts) 	<ul style="list-style-type: none"> • Derivative of A (0.5pts) • Finding a root of A (0.5pts) 	<ul style="list-style-type: none"> • Interval $(0, \infty)$ specified (0.25pts) • Stating that $A' < 0$ when... and $A' > 0$ when... (0.5pts) • Stating that Global Interval Method tells us <u>there's</u> an absolute maximum at... (0.25pts) • Note: Please <u>don't</u> deduct points if they don't specify that A is differentiable on $(0, \infty)$ and don't say that there's a local minimum. 	<ul style="list-style-type: none"> • Solving for height (0.5pts) • Units (0.25pts) • Exact answer or rounded correctly (0.25pts)

Thank You

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