Stats Weekly Application to Games 1
Texas Hold’em – the perils of the inside straight draw

Rules: Each player gets two cards face down
Five cards are dealt face up, shared between all players
First three are called the ‘flop’, fourth is the ‘turn’, fifth is the ‘river’
Each player tries to make the best poker hand from the 7 cards (5 shared, 2 own)

Inside Straight Draw
Say you have 7 9, flop is 6 10 Ace. An 8 would complete your straight!
Prob. of 8 on turn? On river if not on turn? In next two cards?

IMPORTANT: treat the cards other players have in their hands (and any cards the dealer has ‘burned’) as being still in the deck, because we don’t know what they are so they don’t affect the probabilities. If we thought we knew what someone was holding (bad poker face or they showed their cards), we would use that information.

Prob. of 8 on turn: 4/47 ~ 8.5%
Prob. of 8 on river given it didn’t occur on turn: 4/46 ~ 8.7%
Prob. of 8 in next two cards?
CANNOT just add up above probabilities! That will double-count. Need 3 cases:

<table>
<thead>
<tr>
<th>Turn</th>
<th>River</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>not 8</td>
<td>(4/47)x(43/46) ** using the multiplication rule</td>
</tr>
<tr>
<td>not 8</td>
<td>8</td>
<td>(43/47)x(4/46)</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>(4/47)x(3/46)</td>
</tr>
</tbody>
</table>

Now add up (**using the addition rule) these three cases ~ 16.5%

Open-ended Straight Draw
Now, what if you have 7 9 and flop is 6 8 King. A 5 OR a 10 would give you a straight!

Prob. of 5 or 10 on turn: 8/47 ~ 17%
Prob. of 5 or 10 on river given it didn’t occur on turn: 8/46 ~ 17.4%
Prob. of 5 or 10 in next two cards? Again need 3 cases:

<table>
<thead>
<tr>
<th>Turn</th>
<th>River</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 or 10</td>
<td>not 5 or 10</td>
<td>(8/47)x(39/46)</td>
</tr>
<tr>
<td>not 5 or 10</td>
<td>5 or 10</td>
<td>(39/47)x(8/46)</td>
</tr>
<tr>
<td>5 or 10</td>
<td>5 or 10</td>
<td>(8/47)x(7/46)</td>
</tr>
</tbody>
</table>

Now add up these three cases ~ 31.5%

What if you need two runners to make the straight?
You have 7 9. Flop is 2 5 King. You need 6 AND 8 to complete your straight.
Prob. you make it? Only two ways this can happen:

<table>
<thead>
<tr>
<th>Turn</th>
<th>River</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>8</td>
<td>(4/47)x(4/46)</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
<td>(4/47)x(4/46)</td>
</tr>
</tbody>
</table>

Now add up these two cases ~ 1.48%
Stats Weekly Application to Games 2
Magic: The Gathering

(Simplified) Rules: You have a deck of 60 cards (e.g. Land, Artifact, Creature, Enchantment, etc)
You have a deck of 60 cards (e.g. Land, Artifact, Creature, Enchantment, etc)
Your deck can contain up to 4 copies of any particular card
You draw a hand of 7 cards
Each turn you: draw one card (except the first turn),
can play a Land card if you have one,
can attack or cast spells (costs mana which Land cards give)
discard down to 7 cards (cards do NOT go back in deck)

First let’s find the probability of drawing one particular card.

If you have only one copy of it,

\[
P(\text{get it on turn 1}) = 1^* \binom{59}{6}/\binom{60}{7} = (59!/6!53!)/(60!/7!53!) = 7/60 = 11.67\% 
\]

(or logically, card has 1/60 chance to come up with each card you draw, and you get 7 draws)
Similarly, \(P(\text{get it by turn 2}) = 8/60 = 13.33\%\), \(P(\text{get it by turn 3}) = 9/60 = 15\%\), etc

But what if you have more than one copy of it?

If you have two copies,

\[
P(\text{get it (at least one) on turn 1}) = 1 - P(\text{don’t get any on turn 1}) = 1 - \binom{58}{7}/\binom{60}{7} 
= 1 - (58!/7!51!)/(60!/7!53!) = 1 - 53*52/(60*59) = 22.15\% 
\]

Similarly, \(P(\text{get at least one by turn 2}) = 1 - \binom{58}{8}/\binom{60}{8} = 1 - 52*51/(60*59) = 25.08\%\), etc

If you have 4 copies (the maximum you are allowed),

\[
P(\text{get at least one on turn 1}) = 1 - \binom{56}{7}/\binom{60}{7} = 39.95\% 
\]

\[
P(\text{get at least one by turn 2}) = 1 - \binom{56}{8}/\binom{60}{8} = 44.48\% 
\]

And the next probabilities are:

<table>
<thead>
<tr>
<th>Turn</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prob of getting at least one by then</td>
<td>48.8%</td>
<td>52.8%</td>
<td>56.6%</td>
<td>60.1%</td>
<td>63.4%</td>
<td>66.5%</td>
<td>69.4%</td>
<td>72.2%</td>
</tr>
</tbody>
</table>

So there is a >50% chance of getting at least one of a card you have 4 copies of in the first four turns of the game.

How can we use this to decide how many Land cards to put into a deck?

We want to have enough Land cards because they provide mana to cast spells.
But we don’t want to have too many, or there will not be room in the deck for other good cards. It would be a good idea to have a high probability of having at least 4 Land cards drawn by the end of turn 4.

Let’s suppose you have \( N > 4 \) Land cards in your deck of 60. (That means there are \( 60 - N \) non-Land). What is the probability you draw at least 4 of them in your first 4 turns?

\[
P(\text{at least 4 Land by turn 4}) = P(\text{at least 4 Land in 10 cards})
\]
\[
= 1 - P(0 \text{ Land}) - P(1 \text{ Land}) - P(2 \text{ Land}) - P(3 \text{ Land})
\]

\( P(0 \text{ Land}) \) is straightforward and is \( \frac{60 - N}{10} \) since all 10 cards must be non-Land

\( P(1 \text{ Land}) \) is a bit trickier. We want to choose exactly 1 of the \( N \) Land cards AND exactly 9 of the \( 60 - N \) non-Land cards. There are \( \binom{N}{1} \times \binom{60 - N}{9} \) ways to do that. So the probability is

\[
\frac{N}{1} \times \frac{60 - N}{9} / \frac{60}{10}
\]

Similarly, \( P(2 \text{ Land}) = \binom{N}{2} \times \frac{60 - N}{8} / \frac{60}{10} \) and \( P(3 \text{ Land}) = \binom{N}{3} \times \frac{60 - N}{7} / \frac{60}{10} \)

Putting it all together, \( P(\text{at least 4 Land by turn 4}) \)
\[
= 1 - \left[ \frac{60 - N}{10} + \frac{N}{1} \times \frac{60 - N}{9} + \frac{N}{2} \times \frac{60 - N}{8} + \frac{N}{3} \times \frac{60 - N}{7} \right] / \frac{60}{10}
\]

This is a pretty complicated function, but we can evaluate it for various values of \( N \) without too much trouble and graph it:

If you want an 80% chance of drawing at least 4 land in your first 4 turns, \( N = 28 \) or 29 will do it. If you want to be a bit riskier and have only ~65% chance, \( N = 24 \) or 25 is around there.
Stats Weekly Application to Games 3
Who is the Spy? – The Resistance

Rules: Each player is (secretly) either a Spy or a Rebel
A subset of players goes on a “mission” and each player secretly plays a card to have the mission pass or fail (Rebels play pass cards, Spies may choose to play pass or fail)
If three missions pass, the Rebels win. If three missions fail, the Spies win.

The number of Spies depends on the number of players, but for this example we will use 6 players (A, B, C, D, E, F) with 2 spies among them.

What is the probability that a random player is a Spy?

Before we have any information, we can calculate
P(A Spy) = # of ways to have A be the Spy / # of ways there can be 2 Spies among 6 players

\[
P(A \text{ Spy}) = \frac{\binom{5}{1} \times \binom{6}{2}}{15} = \frac{5}{15} = 0.3333
\]

This probability is the same for all players, so P(B Spy) = P(C Spy) = … = P(F Spy) = 0.3333.
Note that the six probabilities add up to 2, not 1, because there are 2 Spies.

Now suppose a mission with 3 players (A, B, and C) has failed with one fail card. What does that tell us?

We now know at least one of A, B, and C is a Spy. Given this information, we can calculate

\[
P(A \text{ Spy} \mid \geq 1 \text{ of } ABC \text{ Spy}) = \frac{P(A \text{ Spy} \cap \geq 1 \text{ of } ABC \text{ Spy})}{P(\geq 1 \text{ of } ABC \text{ Spy})}
\]

If “A Spy” is true, then “\geq 1 of ABC Spy” is automatically true, so the intersection is “A Spy”

\[
P(\geq 1 \text{ of } ABC \text{ Spy}) = 1 - P(\text{none of ABC are Spies}) = 1 - \frac{3}{12} = 1 - \frac{3}{15} = 12/15
\]

So P(A Spy \mid \geq 1 of ABC Spy) = (5/15) / (12/15) = 0.41666
And the same would be true for B and C.

What about the players that didn’t go on that mission? (D, E, and F)

\[
P(D \text{ Spy} \mid \geq 1 \text{ of } ABC \text{ Spy}) = \frac{P(D \text{ Spy} \cap \geq 1 \text{ of } ABC \text{ Spy})}{P(\geq 1 \text{ of } ABC \text{ Spy})}
\]

If we want “D Spy” and “\geq 1 of ABC Spy” to both be true, we need D and exactly one of A, B, C to be Spies, so

\[
P(D \text{ Spy} \cap \geq 1 \text{ of } ABC \text{ Spy}) = 1 \times \frac{\binom{3}{1} \times \binom{6}{2}}{15} = \frac{3}{15}
\]

So P(D Spy \mid \geq 1 of ABC Spy) = (3/15) / (12/15) = 0.25
And the same would be true for E and F.

Notice again that the six probabilities still add up to 2, as they must.

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Now suppose another mission (with A, D, and E) has ALSO failed with one fail card. What can we say now?

We have the info that at least one of A, D, E is a Spy (in addition to at least one of A, B, C)

\[
P(A \text{ Spy} \mid \geq 1 \text{ of } ABC \text{ Spy} \cap \geq 1 \text{ of } ADE \text{ Spy})
\]
\[
= P(A \text{ Spy} \cap \geq 1 \text{ of } ABC \text{ Spy} \cap \geq 1 \text{ of } ADE \text{ Spy}) / P(\geq 1 \text{ of } ABC \text{ Spy} \cap \geq 1 \text{ of } ADE \text{ Spy})
\]

For the numerator, again “A Spy” implies the other two are true so the intersection is “A Spy”

For the denominator, the events are not independent!
There are two ways this can happen. Either A is a Spy and any one other player is a Spy, or one of B or C is a Spy and one of D or E is a Spy.
So the probability is

\[
\frac{5 \times \binom{6}{1}/\binom{2}{1} + \binom{2}{1} \times \binom{2}{1}/\binom{6}{2}}{9/15} = \frac{5}{15} + \frac{4}{15} = \frac{9}{15}
\]

So \(P(A \text{ Spy} \mid \geq 1 \text{ of } ABC \text{ Spy} \cap \geq 1 \text{ of } ADE \text{ Spy}) = (5/15) / (9/15) = 0.5555\)

A is looking pretty suspicious!

What about the other 4 players who have been on one of the 2 failed missions? (B, C, D, E)

\[
P(B \text{ Spy} \mid \geq 1 \text{ of } ABC \text{ Spy} \cap \geq 1 \text{ of } ADE \text{ Spy})
\]
\[
= P(B \text{ Spy} \cap \geq 1 \text{ of } ABC \text{ Spy} \cap \geq 1 \text{ of } ADE \text{ Spy}) / P(\geq 1 \text{ of } ABC \text{ Spy} \cap \geq 1 \text{ of } ADE \text{ Spy})
\]

For the numerator, we must have B and one of A, D, or E being the Spies, which has probability

\[
\frac{3 \times \binom{6}{1}/\binom{2}{1}}{9/15} = \frac{3}{15}
\]

The denominator is the same as before, 9/15.
So \(P(B \text{ Spy} \mid \geq 1 \text{ of } ABC \text{ Spy} \cap \geq 1 \text{ of } ADE \text{ Spy}) = (3/15) / (9/15) = 0.3333\)
And the same would be true for C, D, and E.

Weirdly, the same probability as they had before we knew any information!

Finally, what about F, who has never been on a failed mission?

\[
P(F \text{ Spy} \mid \geq 1 \text{ of } ABC \text{ Spy} \cap \geq 1 \text{ of } ADE \text{ Spy})
\]
\[
= P(F \text{ Spy} \cap \geq 1 \text{ of } ABC \text{ Spy} \cap \geq 1 \text{ of } ADE \text{ Spy}) / P(\geq 1 \text{ of } ABC \text{ Spy} \cap \geq 1 \text{ of } ADE \text{ Spy})
\]

For the numerator, there is only one way that all events can occur: if the Spies are A and F. So the probability is \(1/\binom{6}{2}\). The denominator is the same as before, 9/15.
So \(P(F \text{ Spy} \mid \geq 1 \text{ of } ABC \text{ Spy} \cap \geq 1 \text{ of } ADE \text{ Spy}) = (1/15) / (9/15) = 0.1111\)
Still a possibility that F is a Spy, but very unlikely now.

Notice that if you are playing the game and you know you are (or are not) a Spy, you can include this information in your calculations by treating it like a game with one fewer player.
Stats Weekly Application to Games 4
Beat the House – Card Counting in BlackJack

Rules: You get 2 cards, and so does the “dealer”
10, J, Q, and K are worth 10, A is worth 1 or 11
You can ask for as many more cards as you want, one at a time.
You want to have a higher total than the dealer, without going over 21.
A combination of one A and one card worth 10 (10, J, Q, K) is called a BlackJack

There can be multiple players, but they are all playing against the dealer, not against each other, so we will just consider one player.

Suppose you are playing with one standard deck (52 cards)

20 of the 52 cards are high cards (10, J, Q, K, A) which you want to get. A combination of any two of those would be good for you. Assume they are randomly spread throughout the deck.

At the start, the probability you get two of those cards is \( \frac{20}{2} \times \frac{52}{2} = 190/1326 = 0.143 \)
(Why? Let X be the number of high cards you get. \( X \sim \text{Hyp}(N=52, r=20, n=2) \) and you want \( P(X=2) \) which is \( \frac{20}{2} \times \frac{32}{0} \times \frac{52}{2} \).)

But as you continue to play more rounds, you get better information about what cards are remaining in the deck. If you keep track of this information, this is called “card counting.”

Suppose 26 cards have been played, so you know there are 26 cards left: \( r \) high cards and 26 - \( r \) low cards. I.e. you have been counting how many high cards that have been played so far.

Now the number of high cards you get is \( X \sim \text{Hyp}(N=26, r=r, n=2) \) and the probability you get two high cards is \( \frac{r}{2} \times \frac{26-r}{0} \times \frac{26}{2} \). We can evaluate this probability for different values of \( r \):

<table>
<thead>
<tr>
<th>High cards played so far</th>
<th>13</th>
<th>12</th>
<th>11</th>
<th>10</th>
<th>9</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>r</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>18</td>
</tr>
<tr>
<td>( P(X=2) )</td>
<td>0.06</td>
<td>0.09</td>
<td>0.11</td>
<td>0.14</td>
<td>0.17</td>
<td>0.20</td>
<td>0.24</td>
<td>0.28</td>
<td>0.32</td>
<td>0.37</td>
<td>0.42</td>
<td>0.47</td>
</tr>
</tbody>
</table>

Note: \( r \) could be as low as 0 or as high as 20 but it is extremely unlikely so I have left those probabilities off.

So having that information gives you a lot of power. If there are lots of high cards left (relative to the remaining cards in the deck), you are much more likely to get a good hand, and so you can bet more money accordingly.

That’s why BlackJack is not played with one deck. It’s played with at least 4 decks. But why does that matter?
Suppose you are playing with 4 decks. So there are 80 high cards in the 208 combined cards.

Again assuming they are equally scattered in the deck, the probability you get 2 high cards at the start is $\binom{80}{2}/\binom{208}{2} = 3160/21528 = 0.147$, very close to what it was before with one deck.

Now again assume 26 cards have been played, and there are $r$ high cards of the 208-26 = 182 cards left. So the number of high cards we get is $X \sim \text{Hyp}(N=182, r=r, n=2)$. If we have been counting the number of high cards played so far, we know $r$, so we get the following for $P(X=2)$:

<table>
<thead>
<tr>
<th>High cards played so far</th>
<th>13</th>
<th>12</th>
<th>11</th>
<th>10</th>
<th>9</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>67</td>
<td>68</td>
<td>69</td>
<td>70</td>
<td>71</td>
<td>72</td>
<td>73</td>
<td>74</td>
<td>75</td>
<td>76</td>
<td>77</td>
<td>78</td>
</tr>
<tr>
<td>$P(X=2)$</td>
<td>0.13</td>
<td>0.14</td>
<td>0.14</td>
<td>0.15</td>
<td>0.15</td>
<td>0.16</td>
<td>0.16</td>
<td>0.16</td>
<td>0.17</td>
<td>0.17</td>
<td>0.18</td>
<td>0.18</td>
</tr>
</tbody>
</table>

The exact same information (knowing how many high cards have been played out of the first 26) gives us hardly any difference in the chance our hand has two high cards! So it’s harder to use that information to adjust your betting strategy.

Of course, as we get closer to the end of the 4 decks of cards, we have more information and we can use it if we count cards. That’s why the dealer usually stops and reshuffles before getting too close to the end of the 4 decks, before the information becomes too powerful.

**What if we could play with an infinite number of decks?**

If we have an infinite number of standard decks, then the probability that any card is a high card is just $20/52 = 0.3846$. And importantly, the cards are all independent.

So we don’t need a Hypergeometric distribution anymore, we can use the Binomial. If $X$ is the number of your cards that are high cards, $X \sim \text{Bin}(n=2, p=0.3846)$.

So $P(X=2) = \binom{2}{2}(0.3846)^2(1-0.3846)^0 = 0.148$. Again, pretty close to the 4-deck case.

(Why is it so close? When $N$ is large and $n$ is small, we can approximate a Hypergeometric with a Binomial pretty well. $N=208$ and $n=2$ fits this nicely.)

But if we have an infinite number of decks, then NO MATTER WHAT happens in the first 26 cards (even if all of them or none of them are high cards), the probability of you having two high cards never changes. It’s always 0.148. So you never have any information you can use to adjust your strategy.
Rules: You find “wild” Pokémon in the grass randomly, some species are rarer than others.
You try to catch them with a Pokéball as follows:
   - The species has a catch rate a (a number between 1 and 255).
   - And a shake probability b (a function of a, between 0 and 65535).
   - Four “shake checks” are performed, and the Pokémon is caught if it passes all 4.

First let’s look at the probability of catching a Pokémon once you find one

A “shake check” involves the computer generating a random number between 0 and 65535, and if that number is greater than or equal to b, the shake check fails.

The random number is $X \sim \text{Uniform}(0, 65535)$ so we can find

$$P(\text{check fails}) = P(X \geq b) = 1 - P(X \leq b) = 1 - F(b-1) = (b-1-0+1)/(65535-0+1) = 1 - b/65536$$

and thus $P(\text{check passes}) = b/65536$.

So $P(\text{catch Pokémon}) = P(\text{pass 4 checks}) = (b/65536)^4$.

If you don’t catch it, the number of checks passed before failing one is represented visually by the number of times the Pokéball shakes before the Pokémon breaks free.

So to find the number of shakes, we can find the distribution of the number of shake check passes before the first shake check fail.

This kind of sounds like a Geometric distribution, with Success = “fail a shake check”, except that it doesn’t go on forever – if you get 4 “passes”, it just stops.

But we can still use the Geometric distribution for the probabilities of having 0, 1, 2, or 3 shakes before the Pokémon escapes, as follows:

$$P(\text{no shakes}) = P(\text{fail first check}) = 1 - b/65536$$
$$P(\text{1 shake}) = P(\text{pass first, fail second}) = (b/65536)(1 - b/65536)$$
$$P(\text{2 shakes}) = P(\text{pass first two, fail third}) = (b/65536)^2(1 - b/65536)$$
$$P(\text{3 shakes}) = P(\text{pass first three, fail fourth}) = (b/65536)^3(1 - b/65536)$$

And we know from before $P(\text{catch}) = (b/65536)^4$.

You can verify that these 5 probabilities add up to 1.

Obviously the probabilities depend on the value of b (which depends on a, which depends on lots of things like the species and health and what kind of Pokéball you are using) but here are some sample probabilities:

<table>
<thead>
<tr>
<th>Value of b (shake prob)</th>
<th>0 shakes, just escapes</th>
<th>1 shake then escapes</th>
<th>2 shakes then escapes</th>
<th>3 shakes then escapes</th>
<th>Caught!</th>
</tr>
</thead>
<tbody>
<tr>
<td>65535</td>
<td>0.0015%</td>
<td>0.0015%</td>
<td>0.0015%</td>
<td>0.0015%</td>
<td>99.994%</td>
</tr>
<tr>
<td>60000</td>
<td>8.45%</td>
<td>7.73%</td>
<td>7.08%</td>
<td>6.48%</td>
<td>70.26%</td>
</tr>
<tr>
<td>50000</td>
<td>23.71%</td>
<td>18.09%</td>
<td>13.80%</td>
<td>10.53%</td>
<td>33.88%</td>
</tr>
<tr>
<td>20000</td>
<td>69.48%</td>
<td>21.20%</td>
<td>6.47%</td>
<td>1.97%</td>
<td>0.87%</td>
</tr>
</tbody>
</table>

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The other thing that’s interesting is the process of finding wild Pokémon in the first place

What process do we know that has events occurring randomly throughout time/space? The Poisson process!

We can think about finding wild Pokémon of different species in two ways:
1. There is a Poisson process with a single rate $\lambda$ for just finding a Pokémon at all, and then once one is found, the species is randomly determined according to some probability distribution (higher probability of being a common species, lower for rarer species)
2. There is a separate Poisson process for each species of Pokémon, each with its own rate $\lambda_i$, where the rate is higher if it’s a more common species and lower for rarer ones.

The really neat thing is that both of these models are exactly the same! This is sometimes called the “thinning” property of the Poisson process, and is discussed in more detail in STAT 333 (or 334) and 433.

Let’s look at an example.

Say you know that a Pokémon is encountered approximately once per 2 minutes of game time. The rate would be $\lambda = \frac{1}{2}$.
The chance that a Pokémon is “Shiny” is approximately 1/8192.
So then a Shiny Pokémon Poisson process would have rate proportional to the rate of finding a Pokémon and the probability of it being Shiny: $\lambda_i = (1/2)*(1/8192) = 1/16384$. (Believe me.)

What is the probability of finding at least one Shiny Pokémon in 12 hours of game time?
Let $X = \#$ Shiny found in 12 hours
$X \sim \text{Poi}(1/16384 \times 12 \times 60) = \text{Poi}(0.0439)$
We want $P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-0.0439} = 4.3\%$

It’s super unlikely!
Stats Weekly Application to Games 6
Items and Rank Progression in League of Legends (LOL)

Rules: Your attacks do damage to enemy champions
You can purchase items which can increase either your attack speed or attack damage
Some items have a chance of a “critical strike” (the “crit chance”) which doubles damage

Suppose without any modifiers, your attack does 100 damage (with a 1% crit chance) and you can attack once per second.

We can find the mean and standard deviation of the total damage you can do in 15 seconds.
W = damage on one attack = \{100 with probability 0.99
\}
\{200 with probability 0.01
E[W] = 100*0.99 + 200*0.01 = 101 and in 15 sec you can do 15 attacks, so 15E[W] = 1515
E[W^2] = 100^2*0.99 + 200^2*0.01 = 10300 so Var(W) = 10300 – (101)^2 = 99, and since attacks are independent of each other, 15Var(W) = 1485 (SD = 38.5)

Now suppose you can purchase one of the following items:
Brawler’s Gloves: +8% crit chance
Long Sword: +10 attack damage
Dagger: +15% attack speed

Recalculating the mean and variance under each scenario:
X = damage with Brawler’s Gloves = \{100 with probability 0.91
\}
\{200 with probability 0.09
E[X] = 100*0.91 + 200*0.09 = 109 and in 15 sec you can do 15 attacks, so 15E[X] = 1635
E[X^2] = 100^2*0.91 + 200^2*0.09=12700 so Var(X) = 2499, and 15Var(X) = 37485 (SD = 193.6)

Y = damage with Long Sword = \{110 with probability 0.99
\}
\{220 with probability 0.01
E[Y] = 110*0.99 + 220*0.01 = 111.1 and in 15 sec you can do 15 attacks, so 15E[Y] = 1666.5
E[Y^2] = 110^2*0.99 + 220^2*0.01=12463 so Var(Y) = 2262, and 15Var(Y) = 33930 (SD = 184.2)

Z = damage with Dagger = \{100 with probability 0.99
(Z is the same as W)
\}
\{200 with probability 0.01
E[Z] = 101 as before, but in 15 sec you can now do 17.25 attacks, so 17E[Z] = 1717
Similarly, Var(Z) = 99 as before, so 17Var(Z) = 1683 (SD = 41.0)

Notice that the three means are all fairly close to one another, but the variances are way off. Not surprisingly, increasing the crit chance has the highest variance, and just allowing more attacks with the same current stats has the lowest variance.

There are items that increase more than one thing in combination, but a similar calculation could be done. Items also have different costs, so that could be factored in to your calculations to determine the optimal way to spend your gold.

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League progression: You start in Bronze V. (Bronze tier, division V)
Once you get 100 League Points (more points are awarded for beating players ranked higher than you), you then need to win 2 out of 3 promotion games to get promoted to Bronze IV.
This continues (get 100 LP, get promoted by winning 2 of 3)
From Bronze I, to get promoted to Silver V you need to win 3 out of 5 promotion games.
Then there is Gold V - I, Platinum V - I, and Diamond V - I.

First a note about the ranking system

It is a modification of the Elo system used for ranking chess players. Your MMR (Match Making Ranking) goes up and down based on your wins/losses, as well as the MMRs of your opponents.

For now let’s suppose that your MMR and your skill increase proportionally to the skills and MMRs of your opponents. With that in mind, the probability that you win any single game would be around 0.5, and you would gain LP consistently for wins and lose LP consistently for losses. (In practice this is not the case; if you have higher MMR than a set amount for your division you gain more LP per game won and lose less LP per game lost, and the reverse.)

Progressing through a division

You start at the bottom with 0 LP. Each time you win a game your LP goes up (let’s assume) a uniform amount between 1 and 25. But if you lose, your LP goes down between -1 and -25.
You only get “demoted” if you lose a bunch of games in a row while at 0 LP, so let’s assume that won’t happen.

If you have a 50% chance of winning each game, your expected LP gain is
\[ P(\text{win}) \times E[\text{LP if you win}] + P(\text{lose}) \times E[\text{LP if you lose}] \]
\[ = 0.5 * 13 + 0.5 * (-13) \quad \text{since each one is a uniform random variable centred at 13 or -13} \]
\[ = 0. \]

Well that sucks. If you are in the perfect division for you (equally matched skill-wise) then you are expected to stay there, on average. What about all the n00bs you want to pwn?

Now assume you are placed below where you belong skill-wise, so your win percentage is actually 60%, and you keep improving so it stays 60% consistently.

Now the expected LP gain per game is 0.6*13 + 0.4*(-13) = +2.6. So it would take an average of 100/2.6 = 38.46 games to get to 100 LP, assuming you don’t get demoted.

Then you have to win your promotion matches. This is just a “best 2 out of 3” series, so the probability you win that is:
\[ P(\text{Win 1st, Win 2nd}) + P(\text{Win 1st, Lose 2nd, Win 3rd}) + P(\text{Lose 1st, Win 2nd, Win 3rd}) \]
\[ = 0.6*0.6 + 0.6*0.4*0.6 + 0.4*0.6*0.6 = 0.648 \]
Assuming attempts at being promoted are independent, it would take an average of 5.16 games to get promoted. If you want to find out how I calculated that number, try taking STAT 330, 333, or 334. It involves conditional random variables.

Getting promoted from Bronze I to Silver V (or any other change in tier) is a “best 3 out of 5” series, with probability of winning given by:

\[ P(\text{Win first 3}) + 3P(\text{Win 2 of first 3, Win 4th}) + 6P(\text{Win 2 of first 4, Win 5th}) = 0.68256 \]

So it would take an average 7.71 games to get promoted. (Again, calculated using conditioning.)

So all together, on average*, to get from Bronze V up to Diamond I it would take:

\[
\begin{align*}
24 \times 38.46 & = \quad 923 \text{ games to progress through each division} \\
+ \quad 19 \times 5.16 & = \quad 98 \text{ games to get promoted between divisions within each tier} \\
+ \quad 4 \times 7.71 & = \quad 31 \text{ games to get promoted between tiers} \\
& = 1052 \text{ games total.}
\end{align*}
\]

What are you doing this weekend? 😊

* This is just the average. The variance is still huge around this number!
Stats Weekly Application to Games 7
Cal Me a Liar - Liars’ Dice

Rules: Each player rolls 5 fair 6-sided dice (they can see their own but no one else’s)
1’s are “wild” and can count as any number
Players take turns bidding on the total number of a certain value on all the dice
Each player must either bid higher or challenge the previous bid

Suppose there are 7 players, so 35 dice total

Without looking at any of the dice, the number of 6’s is Bin(35, 1/3)
(p=1/3 because both 6 and 1 count as 6)

So the expected number of 6’s is 35/3 = 11.667 and variance is 35/3*2/3 = 7.778 (std dev 2.789)

But you can see your own dice! Suppose you have 2 6’s

Then the only randomness comes from the 30 dice you cannot see.
Let X = # 6’s on remaining dice. X~Bin(30, 1/3)

So expected number total is 2 + 30/3 = 12 and variance is 30/3*2/3 = 6.667 (std dev 2.582)

What can you bid and be “pretty sure” there are at least that many 6’s? (i.e. so that if you get challenged, you will end up being right).

If you bid “k 6’s”, then the probability there are at least k 6’s is

\[
P(X \geq k - 2) = 1 - P(X \leq k - 3)\]

<table>
<thead>
<tr>
<th>k</th>
<th>Probability of at least k 6’s</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.0000</td>
</tr>
<tr>
<td>3</td>
<td>0.999995</td>
</tr>
<tr>
<td>4</td>
<td>0.999917</td>
</tr>
<tr>
<td>5</td>
<td>0.999349</td>
</tr>
<tr>
<td>6</td>
<td>0.996703</td>
</tr>
<tr>
<td>7</td>
<td>0.98777</td>
</tr>
<tr>
<td>8</td>
<td>0.964546</td>
</tr>
<tr>
<td>9</td>
<td>0.916162</td>
</tr>
<tr>
<td>10</td>
<td>0.833217</td>
</tr>
<tr>
<td>11</td>
<td>0.713984</td>
</tr>
<tr>
<td>12</td>
<td>0.568256</td>
</tr>
<tr>
<td>13</td>
<td>0.41524</td>
</tr>
<tr>
<td>14</td>
<td>0.276136</td>
</tr>
<tr>
<td>15</td>
<td>0.166011</td>
</tr>
<tr>
<td>16</td>
<td>0.089771</td>
</tr>
<tr>
<td>17</td>
<td>0.043482</td>
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<tr>
<td>18</td>
<td>0.018795</td>
</tr>
<tr>
<td>19</td>
<td>0.007223</td>
</tr>
<tr>
<td>20</td>
<td>0.002458</td>
</tr>
<tr>
<td>21</td>
<td>0.000737</td>
</tr>
</tbody>
</table>

So you can be more than 95% confident in a bid of 8 6’s, more than 50% confident in a bid of 12 6’s, and if you’re feeling risky, more than 25% confident in a bid of 14 6’s.

In Pirates of the Caribbean 2 (Dead Man’s Chest), they play 3-player, but without 1’s being wild. The video is here https://youtu.be/8vUkTpzXQZc?t=139 if you want to see it. Let’s see how the game goes:
Will Turner bids 4 5’s when he has 3 of his own. The probability of him being correct is the probability that the 10 dice he can’t see have at least one 5, which is: 
\[ P(X \geq 1) = 1 - (5/6)^{10} = 0.838. \] Not a very bold move. (here \( X \sim \text{Bin}(10, 1/6) \))

Bootstrap Bill ups the bid to 6 3’s (not particularly relevant, although it indicates he probably does not have a lot of 5’s)

Then Davy Jones then bids 7 5’s, having 4 of his own. The probability of him being correct is the probability that the 10 dice he can’t see have at least 3 5’s, which without any info would be:
\[ P(X \geq 3) = 1 - (5/6)^{10} - (10C1)(1/6)(5/6)^9 - (10C2)(1/6)^2(5/6)^8 = 0.225, \] much riskier.
But really he has the information that Will probably has more than an average number of 5’s since he bid them. (If he was telling the truth!)

Will is forced to bid higher or challenge, and he makes the statistically incorrect choice (without any info) to bid higher. From his perspective (with 3 5’s showing) the probability of there being fewer than 7 5’s total (so that challenging is correct) is:
\[ P(X < 4) = (5/6)^{10} + (10C1)(1/6)(5/6)^9 + (10C2)(1/6)^2(5/6)^8 + (10C3)(1/6)^3(5/6)^7 = 0.930 \]
whereas the probability of there being 8 or more 5’s (so that upping the bid is correct) is only:
\[ P(X \geq 5) = 0.015 \] (verify the calculation yourself if you like.)
But again, he has the information that Davy probably has much more than an average number of 5’s since he bid them so high. As it turns out there are exactly 7 5’s total (probability 0.054 from Will’s perspective) which makes things super-dramatic!

Luckily his dad Bill saves him by making a statistically impossible bid (he has no 5’s but bids 12 5’s overall, which is not possible) thus losing on purpose so Will doesn’t lose his soul.

Anyway, you can see how each player’s perspective on the game changes with each bid, even in a simple 3-player game. Have fun lying to each other!
"Into Darkness" is a sci fi short story by Greg Egan about a dangerous phenomenon called *The Intake* which affects an area of a city for an exponentially distributed amount of time. If you're interested in reading the whole story you can borrow it from me or buy it in the collection called "Axiomatic" [https://www.amazon.ca/Axiomatic-Short-Stories-Science-Fiction/dp/1597805408](https://www.amazon.ca/Axiomatic-Short-Stories-Science-Fiction/dp/1597805408)

Here is a short excerpt from the story:

*The Intake's manifestations obey exactly the same statistics as a radioactive nucleus with a half-life of eighteen minutes; seventy-five percent last six minutes or more. I've memorized the probabilities right out to an hour (ten percent.)

A mere ten percent of manifestations last for an hour or more - but of that ten percent, half will still be there eighteen minutes later. For someone inside to ask what the odds are now, he or she must be alive to ask the question, and so the probability curve must start afresh from that moment. History can't harm you; the "chance" of having survived the last x minutes is 100%, once you've done it."

Let's break it down and verify some of the calculations.

*The Intake's manifestations obey exactly the same statistics as a radioactive nucleus with a half-life of eighteen minutes; seventy-five percent last six minutes or more..."

If X = the amount of time The Intake lasts in minutes, X ~ Exp(θ). We are told the half-life is 18 minutes which means P(X > 18) = 0.5. We also know P(X > 18) = 1 - F(18) = 1 - (1 - e^{-18/θ}) = e^{-18/θ}.

So setting these equations equal to each other and solving gives e^{-18/θ} = 0.5 -> θ = -18/ln(0.5) = 25.9685

Now P(X > 6) = 1 - F(6) = 1 - (1 - e^{-6/25.9685}) = 0.7937 (pretty close to the 75% he says)

*I've memorized the probabilities right out to an hour (ten percent)"

P(X > 60) = 1 - F(60) = 1 - (1 - e^{-60/25.9685}) = 0.0992 (nice)

*A mere ten percent of manifestations last for an hour or more - but of that ten percent, half will still be there eighteen minutes later"

If it's already lasted an hour, we want the probability it lasts another 18 minutes or 78 minutes total

P(X > 78 | X > 60) = P(X > 78)/P(X > 60) (the intersection of X > 78 and X > 60 is just X > 78) = (1 - F(78))/(1 - F(60)) = (1 - (1 - e^{-78/25.9685}))/ (1 - (1 - e^{-60/25.9685})) = 0.5

This is just the memoryless property at work! P(X > 78|X > 60) = P(X > 18)

*For someone inside to ask what the odds are now, he or she must be alive to ask the question, and so the probability curve must start afresh from that moment. History can't harm you; the "chance" of having survived the last x minutes is 100%, once you've done it."

That's what being in a memoryless process would feel like - super weird!

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Random Number Generation in pretty much any computer game ever

Rules: Vary by game, but often a random number from a certain distribution is needed
e.g. to determine whether an attack succeeds or fails
to determine the amount of damage an attack does
to determine the value of a reward
to simulate the rolling of a number of dice (sometimes dropping the lowest)

It is fairly trivial for a computer to generate (pseudo-)random U(0,1) numbers

But you don’t usually want random numbers that are U(0,1), you want random numbers that follow a certain distribution.

If the cdf $F(x)$ of the distribution we want is invertible (a 1-1 function, so the inverse exists)

We can use the Inverse Transform method (aka cdf method)

Algorithm: 1. Generate $u \sim U(0,1)$
   2. Let $x = F^{-1}(u)$
   3. $x$ will come from a distribution with cdf $F(x)$

Why? Let’s find the cdf of $X = F^{-1}(U)$.

$$P(X \leq x) = P(F^{-1}(U) \leq x)$$

subbing in the definition of $X$

$$= P(F(F^{-1}(U)) \leq F(x))$$

taking $F(.)$ of both sides, doesn’t change inequality

$$= P(U \leq F(x))$$

since $F(F^{-1}(.)$ is just the identity function

$$= F(x)$$

since the cdf of a U(0,1) is just the argument itself

If the cdf $F(x)$ of the distribution we want has discontinuities

Just define the “inverse” of $F$ at $u$ as the x-coordinate of the point where the line at height $u$ first strikes the graph of the cdf (imagining vertical lines are drawn at the discontinuities).

Algorithm: 1. Generate $u \sim U(0,1)$
   2. Let $x = F^{-1}(u)$ as defined above
   3. $x$ will come from a distribution with cdf $F(x)$

If the range of the distribution we want is bounded on $(a, b)$ and we know the pdf/pf $f(x)$

We can use the pdf/pf method.

Algorithm: 1. Generate $u \sim U(a,b)$ and $v \sim U(0, \max\{f(x)\})$ (if discrete, $u \sim U(a, b+1)$, round down)
   2. Evaluate $f(u)$ (the pdf/pf of the distribution we want, evaluated at $u$)
   3. If $v \leq f(u)$, let $x = u$
   4. If $v > f(u)$, discard both values $u$ and $v$, and start again at step 1
   5. $x$ generated in this way will come from a distribution with pdf/pf $f(x)$
Stats Weekly Application to Games 10
Joint Distributions in Card Games - Hanabi

Rules: The deck has 5 (or 6 but we’ll ignore) suits, with cards 1, 1, 1, 2, 2, 3, 3, 4, 4, 5. (50 total)
You get a hand of 4 (or 5) cards, but you DO NOT see your own cards
You can only see the cards of all the other players (assume 4 players)
Through the game you can give other players clues about the cards they hold
e.g. these two cards are red (point at them), or this card is a 5 (point at it)
The goal is to create 5 piles of cards by suit, going from 1 to 5.

Imagine we have no information at all about anything. Let’s define some variables.

Let X = number of 1’s in your hand, and Y = number of red cards in your hand

Since you have 4 cards, X and Y can be anywhere from 0 to 4. So we can find the joint probability function \( f(x,y) \) of X and Y as follows:

<table>
<thead>
<tr>
<th>y</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0889</td>
<td>0.1707</td>
<td>0.1083</td>
<td>0.0267</td>
<td>0.0021</td>
</tr>
<tr>
<td>1</td>
<td>0.0996</td>
<td>0.1805</td>
<td>0.1153</td>
<td>0.0308</td>
<td>0.0029</td>
</tr>
<tr>
<td>2</td>
<td>0.0345</td>
<td>0.0651</td>
<td>0.0416</td>
<td>0.0104</td>
<td>0.0009</td>
</tr>
<tr>
<td>3</td>
<td>0.0043</td>
<td>0.0095</td>
<td>0.0058</td>
<td>0.0012</td>
<td>0.0001</td>
</tr>
<tr>
<td>4</td>
<td>0.0002</td>
<td>0.0005</td>
<td>0.0003</td>
<td>0.00003</td>
<td>0</td>
</tr>
</tbody>
</table>

Here’s how some of the values in the table are found. All the denominators are the number of ways to choose 4 cards from 50, or \((50C4)\)

First, looking at the \( X=0 \) column, we know we can’t have any of the 15 1’s, so we only care how many of the 4 cards are red (7 available) or non-red (28 available)
\[ f(0,0) = P(\text{no 1’s, no reds}) = (7C0)(28C4)/(50C4) = 0.0889 \]
\[ f(0,1) = P(\text{no 1’s, 1 red}) = (7C1)(28C3)/(50C4) = 0.0996, \text{ similarly for the rest in that column} \]

And similarly for the \( Y=0 \) row, we can’t have any of the 10 red cards, so we only care how many of the 4 cards are 1’s (12 available) or non-1’s (28 available)
\[ f(1,0) = P(\text{one 1, no reds}) = (12C1)(28C3)/(50C4) = 0.1707, \text{ similarly for the rest in that row} \]

On the other end of the scale, if \( X=4 \) then all the cards are 1’s, so we just need to know how many of them are red 1’s (3 available) and non-red 1’s (12 available)
\[ f(4,1) = P(4 1’s, 1 red) = (3C1)(12C3)/(50C4) = 0.0029, \text{ similarly for f}(4,2) \text{ and f}(4,3) \]
BUT \( f(4,4) = 0 \) since there cannot be both 4 1’s and 4 reds (there are only 3 red 1’s in the deck)

Likewise, if \( Y=4 \) all the cards are red, so we just need to know how many are red 1’s (3 available) or red non-1’s (7 available)
\[ f(1,4) = P(\text{one 1, 4 reds}) = (3C1)(7C3)/(50C4) = 0.0005, \text{ similarly for f}(2,4) \text{ and f}(3,4) \]
Finally, in the interior of the table (where both X and Y are between 1 and 3), things are a bit more complicated since it matters if the 1’s are red or not. In the deck there are 3 red 1’s, 7 red non-1’s, 12 non-red 1’s, and 28 neither. Each combination of X and Y needs 2 cases (or if X=Y=2, 3 cases) to find all the ways:

\[
f(1,1) = P(\text{one 1, 1 red}) = P(1 \text{ red 1, 3 neither}) + P(1 \text{ red non-1, 1 non-red 1, 2 neither})
\]
\[
= (3C1)(28C3)/(50C4) + (7C1)(12C1)(28C2)/(50C4) = 0.0427 + 0.1379 = 0.1805
\]

\[
f(1,2) = P(\text{one 1 red, 1 non-red 1, 2 neither}) + P(\text{2 red non-1s, 1 non-red 1, 1 neither})
\]
\[
= (3C1)(7C1)(28C2)/(50C4) + (7C2)(12C1)(28C1)/(50C4) = 0.0345 + 0.0306 = 0.0651
\]

\[
f(1,3) = (3C1)(7C2)(28C1)/(50C4) + (7C3)(12C1)/(50C4) = 0.0095
\]

\[
f(2,1) = P(\text{2 1s, 1 red}) = P(\text{1 red 1, 1 non-red 1, 2 neither}) + P(\text{1 red non-1, 2 non-red 1s, 1 neither})
\]
\[
= (3C1)(12C1)(28C2)/(50C4) + (7C1)(12C2)(28C1)/(50C4) = 0.0651
\]

\[
f(2,2) = P(\text{2 red 1s, 2 neither}) + P(\text{1 of each of the 4 types}) + P(\text{2 red non-1s, 2 non-red 1s})
\]
\[
= (3C2)(28C2)/(50C4) + (3C1)(7C1)(12C1)(28C1)/(50C4) + (7C2)(12C2) = 0.0416
\]

\[
f(2,3) = (3C1)(7C2)(12C1)/(50C4) + (3C2)(7C1)(28C1)/(50C4) = 0.00580
\]

\[
f(3,1) = P(\text{3 1s, 1 red}) = P(\text{1 red 1, 2 non-red 1s, 1 neither}) + P(\text{1 red non-1, 3 non-red 1s})
\]
\[
= (3C1)(12C2)(28C1)/(50C4) + (7C1)(12C3)/(50C4) = 0.0308
\]

\[
f(3,2) = (3C1)(7C1)(12C2)/(50C4) + (3C2)(12C1)(28C1)/(50C4) = 0.0104
\]

\[
f(3,3) = (3C2)(7C1)(12C1)/(50C4) + (3C3)(28C1)/(50C4) = 0.0012
\]

Now our table is complete! What if we just want to focus on one variable at a time?

We can find the marginal distribution of X by summing down the columns, and we get:

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>f_X(x)</td>
<td>0.2274</td>
<td>0.4263</td>
<td>0.2713</td>
<td>0.0691</td>
<td>0.0059</td>
</tr>
</tbody>
</table>

This is just a hypergeometric distribution with N=50, r=15, and n=4, as it should be, since with X we only care about the number of 1’s and there are 15 out of 50.

Similarly we find the marginal distribution of Y by summing across the rows:

<table>
<thead>
<tr>
<th>y</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>f_Y(y)</td>
<td>0.3968</td>
<td>0.4290</td>
<td>0.1524</td>
<td>0.0208</td>
<td>0.0009</td>
</tr>
</tbody>
</table>

Again, this is hypergeometric with N=50, r=10, and n=4.

Are the variables independent?

No. You can check any value and you will find f(x,y) ≠ f_X(x)f_Y(y). Or you can notice the 0 in f(4,4), which means that the ranges of the two variables depend on each other, which clearly means they are not independent.

How could we use this in the game?

We can take into account the information we get from observing other players’ cards. For example, if we see among the other 12 cards: 1 red 1, 2 red non-1s, 2 non-red 1s, and 7 neither, we could update the probabilities accordingly. A few more combinations of X and Y might be made impossible depending on what we can see.
Say you have a portfolio of two stocks, A and B
A has average return $\mu_A$ and standard deviation $\sigma_A$
B has average return $\mu_B$ and standard deviation $\sigma_B$
A and B have correlation coefficient $\rho$

Say you invest $\alpha$ % of your money in A and the rest in B, so $P = \alpha A + (1 - \alpha)B$

We want to know the reward (average return) and risk (standard deviation) of our portfolio $P$. Using the rules for linear combinations of random variables,

$$E[P] = \alpha \mu_A + (1 - \alpha)\mu_B$$
$$\text{Var}(P) = \alpha^2 \sigma_A^2 + (1 - \alpha)^2 \sigma_B^2 + 2\alpha(1 - \alpha)\text{Cov}(A, B)$$
$$= \alpha^2 \sigma_A^2 + (1 - \alpha)^2 \sigma_B^2 + 2\alpha(1 - \alpha)\rho \sigma_A \sigma_B$$ (by definition of $\rho$)

The risk and reward of $P$ depend on the value of $\alpha$ chosen to invest in A, and on the correlation between the two stocks. The graph below is an example where $\mu_A = 2$, $\sigma_A = 0.5$, $\mu_B = 8$, and $\sigma_B = 2.0$. The risk and reward of the portfolio $P$ is plotted for five different values of $\rho$, and for $\alpha$ ranging from $-0.2$ to $1.2$. A value of $\alpha < 0$ corresponds to selling stock A and buying even more stock B, whereas $\alpha > 1$ is the opposite.

We can see that the correlation has a large effect on the risk of the portfolio. If the correlation is $-1$, there is a no-risk portfolio where the line touches the y-axis. Unfortunately, it’s almost impossible to find perfectly negatively correlated stocks in real life. Most would be somewhere in between $-0.5$ and $0.8$ or so.

For any given correlation, we can find the minimum risk portfolio by minimizing $\sigma_P$ with respect to $\alpha$. Then any value of $\alpha$ greater than that minimum value will be an efficient portfolio. (Having a lower value of $\alpha$ would be less reward and more risk – no investor would choose it!) The value of $\alpha$ that an investor chooses would be based on their personal risk preference.
SWAG 12: Player Ranking Systems

As mentioned in the Week 7 SWAG, there are ranking systems for players within a game.

- ELO: The base assumption is that the performance on a game is Normally distributed. If two players have the same score, they each have a 50% chance of winning a game against each other. If the difference in scores is 100 points, the player with the higher score has a 64% chance of winning. With 200 points difference, it is 76%. In this system, you have a score that changes every time you play a game – the winning player takes points from the losing player, according to the expected outcome.

- Glicko: A major issue with ELO is that it is a static score – until you play another game, your score stays the same. Glicko improves on this by incorporating uncertainty about the score through the RD (ratings deviation) – which is just the standard deviation of the Normally distributed score. If a player has a score of \( \mu \) and a ratings deviation of \( \sigma \), then their true score is between \( \mu - 1.96 \sigma \) and \( \mu + 1.96 \sigma \) with 95% confidence. When players play a game, the score and ratings deviation are both adjusted: a player’s score changes by a smaller amount when the player has a low ratings deviation or their opponent has a high ratings deviation.

Why does this matter?
Imagine you are playing someone who has a lower ELO score than you. If you play them in one game, there is a chance they will win. But if you play them repeatedly, the proportion of games you win will approach the true probability of you winning.