How would you quickly convince your curious calculus student (or yourself) that a particular type of limit is an indeterminate form?

You probably routinely use functions $f(x) = ax, a \in \mathbb{R}\{0\}$, and $g(x) = x$, to demonstrate that the limit of the quotient of two functions that both tend to 0, when $x \to 0$, or to $\infty$, when $x \to \infty$, could be whatever.

But what about other indeterminate forms? Like $0^0$, for example?

A student challenges you: “You said that $0^0$ is an indefinite form, but all examples that you showed us ended up with the limit equal 1. How come that $0^0$ is not 1? I actually think that I can prove that it is.”

You may wish to answer student’s question by taking $f(x) = x$ and $g(x) = 0$ and showing that $\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) = 0$, but $\lim_{x \to 0} f(x)g(x) = 1$ and $\lim_{x \to 0} g(x)f(x) = 0$.

Still, your student is not happy and says: “You are taking $g(x) = 0$, so this may be some of those weird special cases. Can you convince me that a limit of the type $0^0$ is really an indeterminate form, i.e., that it could be . . . whatever?”

A simple way to do this is to, for $a \in (0, 1)$, take $f(x) = e^{x^2 \ln a}$ and $g(x) = x^2$. Then, $\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) = 0$ and $\lim_{x \to 0} f(x)g(x) = a$.

To even add a bit more spice and to possibly keep your student in suspense, you may take, for $a > 1$ and $b \in \mathbb{R}\{0\}$, $f(x) = a^{-\sin^{-2}x}$ and $g(x) = x \sin(bx)$. Then, $\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) = 0$ and

$$\lim_{x \to 0} f(x)g(x) = \lim_{x \to 0} a^{\frac{x \sin(bx)}{\sin^2(x)}} = a^{-b},$$

i.e., a limit of the type $0^0$ could be . . . whatever.
But your student is not giving up: “This is convincing, but I’ll prove to you that $0^0 = 1$. Do you accept the binomial formula:

$$
(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k, \ a, b \in \mathbb{R}, n \in \mathbb{N}?
$$

“Of course,” you answer, a bit puzzled. Your student is smiling: “If I take $a = 0$, $b = 1$, and, for example, $n = 2$, the binomial formula gives me: $1 = (0+1)^2 = 0^2 \cdot 1^0 + 2 \cdot 0^1 \cdot 1^1 + 0^0 \cdot 1^2 = 0^0$.”

You are scratching your head, “What has just happened?”

Reference


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