FYMSiC Newsletter
Issue 12, February 2023

A Calculus Vignette

Veselin Jungić, Simon Fraser University

How would you quickly convince your curious calculus student (or yourself) that a particular type of limit is an indeterminate form?

You probably routinely use functions $f(x) = ax$, $a \in \mathbb{R}\backslash\{0\}$, and $g(x) = x$, to demonstrate that the limit of the quotient of two functions that both tend to 0, when $x \longrightarrow 0$, or to ∞, when $x \longrightarrow \infty$, could be whatever.

But what about other indeterminate forms? Like 0^0, for example?

A student challenges you: “You said that 0^0 is an indefinite form, but all examples that you showed us ended up with the limit equal 1. How come that 0^0 is not 1? I actually think that I can prove that it is.”

You may wish to answer student’s question by taking $f(x) = x$ and $g(x) = 0$ and showing that $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} g(x) = 0$, but $\lim\limits_{x\to 0} f(x)^{g(x)} = 1$ and $\lim\limits_{x\to 0} g(x)^{f(x)} = 0$.

Still, your student is not happy and says: “You are taking $g(x) = 0$, so this may be some of those weird special cases. Can you convince me that a limit of the type 0^0 is really an indeterminate form, i.e., that it could be . . . whatever?”

A simple way to do this is to, for $a \in (0, 1)$, take $f(x) = e^{x^{-2}\ln a}$ and $g(x) = x^2$. Then, $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} g(x) = 0$ and $\lim\limits_{x\to 0} f(x)^{g(x)} = a$.

To even add a bit more spice and to possibly keep your student in suspense, you may take, for $a > 1$ and $b \in \mathbb{R}\backslash\{0\}$, $f(x) = a^{-\sin^{-2} x}$ and $g(x) = x\sin(bx)$. Then, $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} g(x) = 0$ and

$$\lim_{x\to 0} f(x)^{g(x)} = \lim_{x\to 0} a^{-\frac{x\sin(bx)}{\sin^2(x)}} = a^{-b},$$

i.e., a limit of the type 0^0 could be . . . whatever.

But your student is not giving up: “This is convincing, but I’ll prove to you that $0^0 = 1$. Do you accept the binomial formula:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,\ a, b \in \mathbb{R},\ n \in \mathbb{N}?”$$

“Of course,” you answer, a bit puzzled. Your student is smiling: “If I take $a = 0$, $b = 1$, and, for example, $n = 2$, the binomial formula gives me: $1 = (0+1)^2 = 0^2 \cdot 1^0 + 2 \cdot 0^1 \cdot 1^1 + 0^0 \cdot 1^2 = 0^0$.”

You are scratching your head, “What has just happened?”

Reference

Huber, M. R., & Rickey, V. F. (2008). What is 0^0?, *MAA Publications: Convergence*. Retrieved from: `https://www.maa.org/press/periodicals/convergence/what-is-00`