A Calculus Vignette

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Student (uncomfortably): This is probably a silly question, but I have to ask...

Instructor (encouragingly): There is no silly question in this class especially when you are learning something new.

Student (quietly): Well, I felt kind of silly in the class today when you were talking about the exponential growth.

Instructor (warily): I am sorry to hear that. Please tell me what I said that made you feel this way.

Student (uneasily): You said that any exponential function with a base greater than one would overtake any polynomial function, regardless how big its degree was. And you talked about it like it was something obvious. I looked around me and felt that I was the only person in the class not getting it. I even punched in my calculator $1.000001^{1000000}$ and got only something like 2.7. There is no way that 1.000001^x would ever be bigger than $x^{1,000,000}$, I think.

Instructor (with relief): First, next time when I say something in the class that you don't understand, please interrupt me and ask for an explanation. That would, actually, help me a lot. You can be sure that if you don't understand something, at least a half of your classmates are in the same boat as you are.

Student (hesitantly): OK, I will.

Instructor (smiling): Secondly, you are right, this fact about the relationship between exponential growth and polynomial growth is not that obvious when you see it for the first time. Would you like me to prove it to you?

Student (excitedly): Yes, please! Just keep it simple.

Instructor (slowly, talking and writing on a piece of paper): I'll show you a proof that I learned when I was a Calculus student. I will show that for any natural number n and any real number a that is greater than one we have that $\lim_{x\to\infty} \frac{x^n}{a^x} = 0$. Would this convince you that any exponential function with the base greater than one grows faster than any polynomial function?

Student (enthusiastically): Yes, it will. If this limit is zero, then this means that the denominator is getting much bigger than the numerator as x is taking very large values. Are you going to use L'Hopital's Rule? That is another one of those things that I have no idea why it works.

Instructor (excitingly): Exactly! And I am not using L'Hopital's Rule. What is kind of funny . . . Because of

$$\frac{x^n}{a^x} = \left(\frac{x}{(\sqrt[n]{a})^x}\right)^n,$$

it is enough to prove that $\lim_{x\to\infty} \frac{x}{a^x} = 0$ for any $a > 1$. How big the power of x is, actually does not matter. Would you like to think about this a bit, or I can continue?

Student (impatiently): Please just continue, I'd like to see where this is going.

Instructor (triumphantly): Let $b > 0$ be such that $a = 1 + b$. Let $x > 2$ and let $c = c(x) \in [0, 1)$ be such that $x - c$ is a natural number. Then, by the Binomial Theorem,

$$a^x \geq (1+b)^{x-c} \geq 1 + b(x-c) + \frac{b^2(x-c)(x-c-1)}{2} > \frac{b^2(x-1)(x-2)}{2}.$$

Since everything is positive, when I take reciprocals and multiply by x, I get

$$0 < \frac{x}{a^x} < \frac{2x}{b^2(x-1)(x-2)}.$$

By the Squeeze Theorem, $\lim_{x\to\infty} \frac{x}{a^x} = 0$.

Student (disappointingly): What? That's all?

Bibliography

Jungić, V., Menz, P., & Pyke, R. (2023). Differential Calculus: Problems And Solutions From Fundamentals To Nuances, *World Scientific*. Retrieved from: `https://doi.org/10.1142/13324`