MATH 137/138:

Calculus 1 and 2 for Honours Mathematics

Brian Forrest

University of Waterloo

Faculty of Math Core Calculus Courses

- Core Calculus courses for students in the Faculty of Mathematics.
- All students have high school Calculus.
- Emphasis is on a deeper understanding of core concepts.

The Courses

MATH 137:

- Chapter 1: Sequence and Convergence
- Chapter 2: Limits and Continuity
- Chapter 3: Derivatives
- Chapter 4: Mean Value Theorem
- Chapter 5: Taylor Polynomials and Taylor's Theorem

MATH 138

- Chapter 1: Integration
- Chapter 2: Techniques of Integration
- Chapter 3: Applications of Integration
- Chapter 4: Differential Equations
- Chapter 5: Numerical Series
- Chapter 6: Power Series

The Courses

MATH 137 and MATH 138

- MATH 137 starts with Sequences in part because it is new.

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- Several computational algorithms are developed such as :
  - Heron's Method
  - Bisection Method
  - Newton's Method
  - Euler's Method

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MATH 137 and MATH 138

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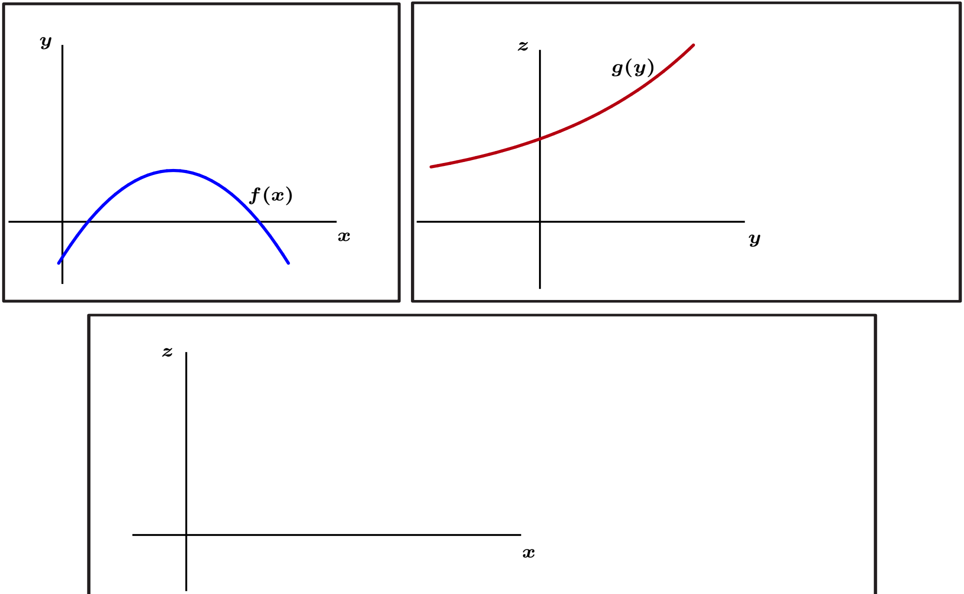
- Several computational algorithms are developed such as :
    - Heron's Method
    - Bisection Method
    - Newton's Method
    - Euler's Method
- "Approximation" is a theme of the course including Linear Approximation, Taylor Polynomials and Big-O notation.

Chain Rule



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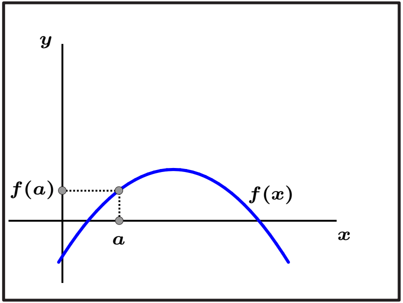

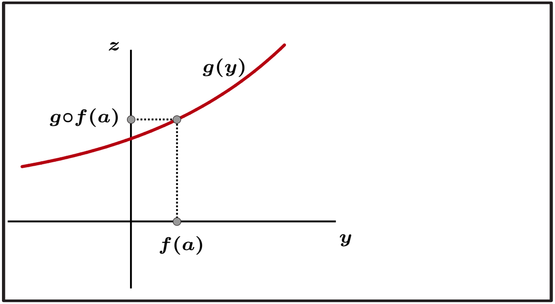

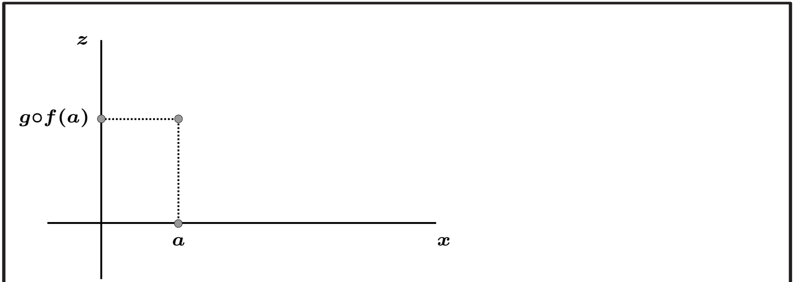

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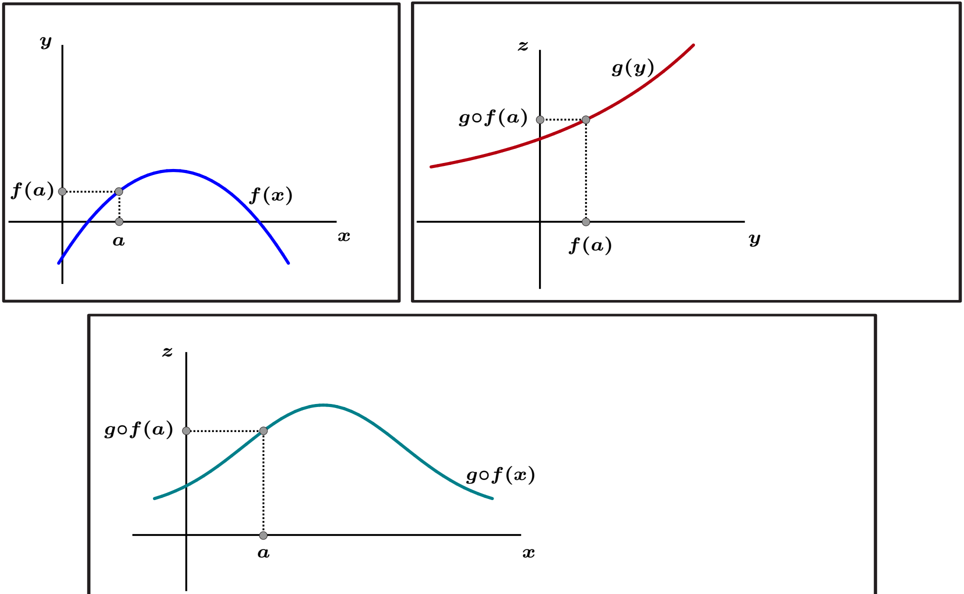

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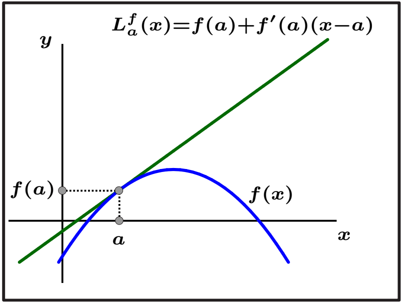

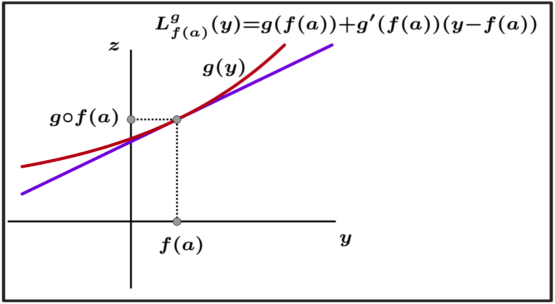

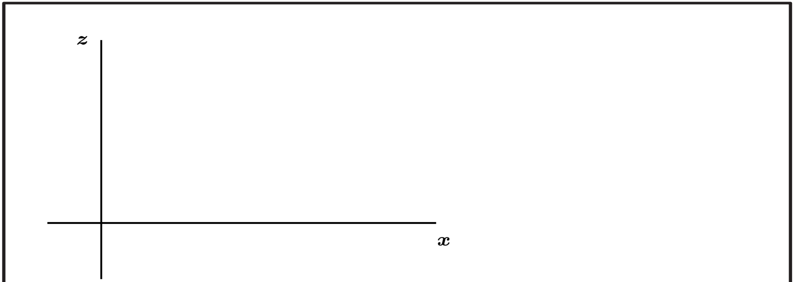

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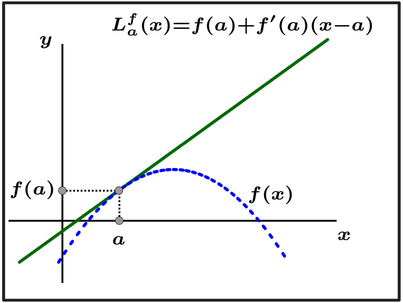

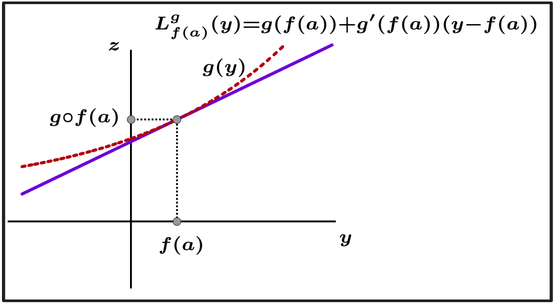

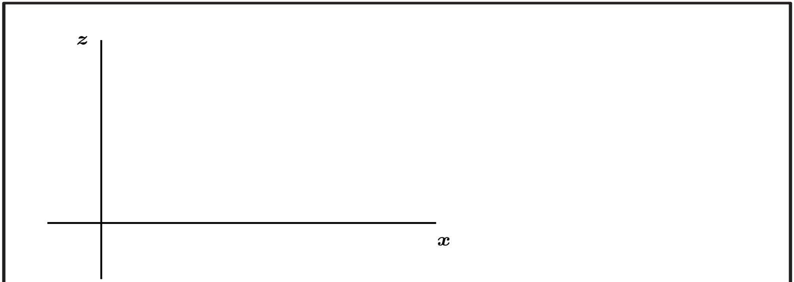

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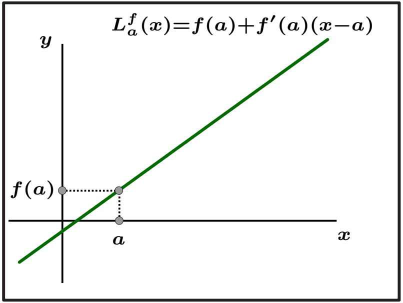

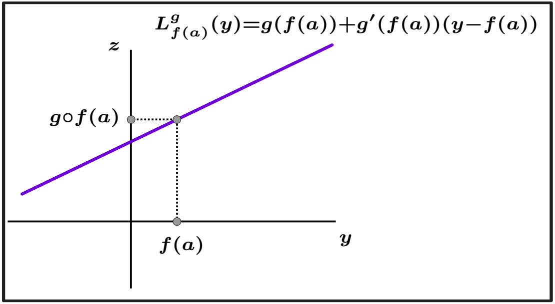

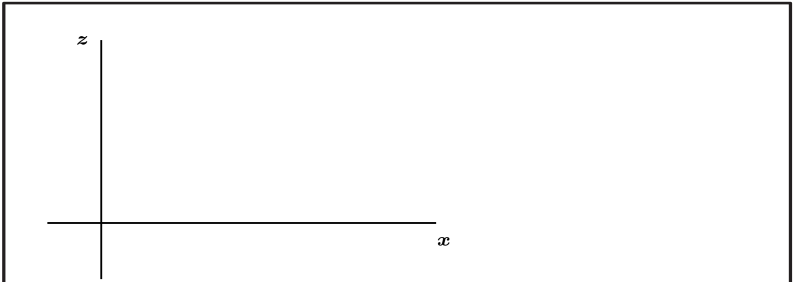

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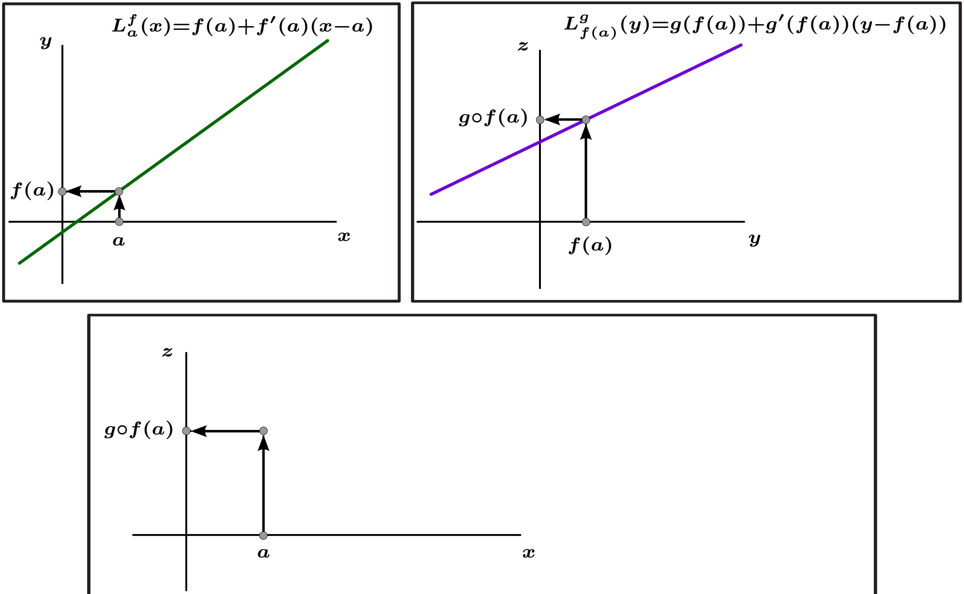

$L_a^f(x) = f(a) + f'(a)(x - a)$
y
$f(a)$
a
x
$L_{f(a)}^g(y) = g(f(a)) + g'(f(a))(y - f(a))$
z
$g \circ f(a)$
$f(a)$
y
z
$g \circ f(a)$
a
x

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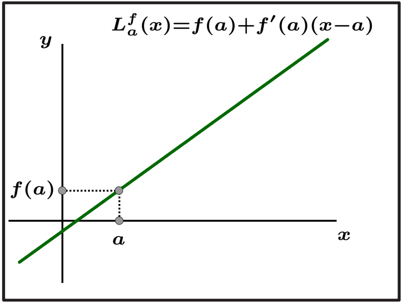
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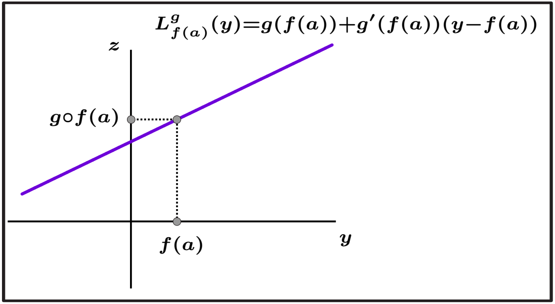
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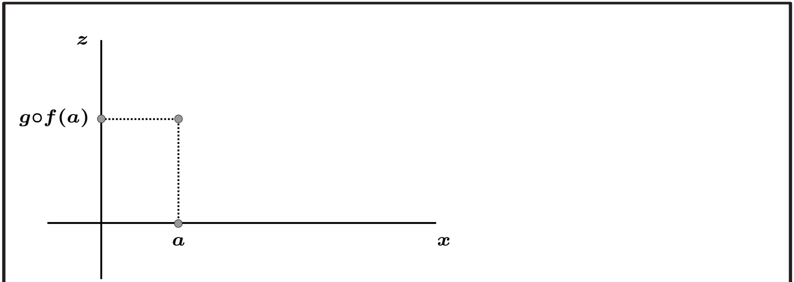
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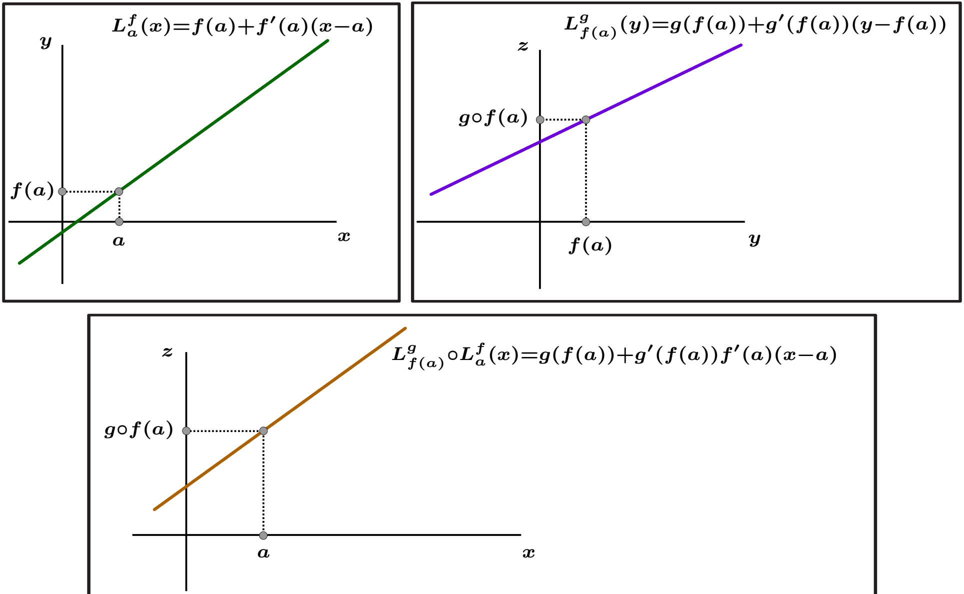

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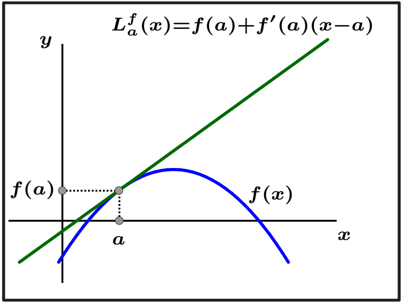
y
$f(a)$
a
x
$L^g_{f(a)}(y) = g(f(a)) + g'(f(a))(y - f(a))$
z
$g \circ f(a)$
$f(a)$
y
$L^g_{f(a)} \circ L_a^f(x) = g(f(a)) + g'(f(a))f'(a)(x-a)$
z
$g \circ f(a)$
a
x

Chain Rule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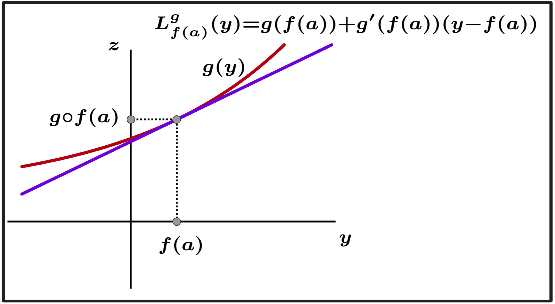
$L_{f(a)}^g(y) = g(f(a)) + g'(f(a))(y - f(a))$
z
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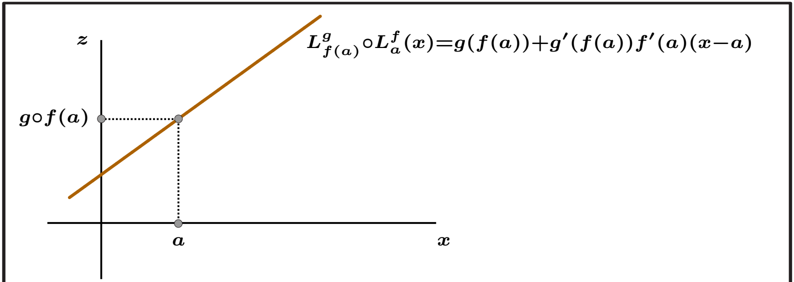
y


$L_{f(a)}^g \circ L_a^f(x) = g(f(a)) + g'(f(a))f'(a)(x-a)$
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x

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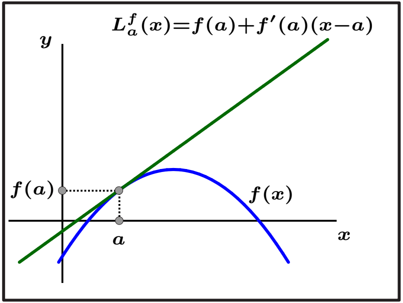

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y
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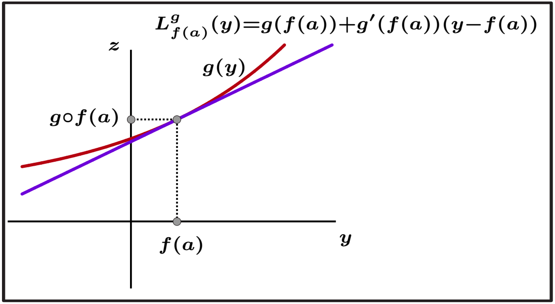

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z
g(y)
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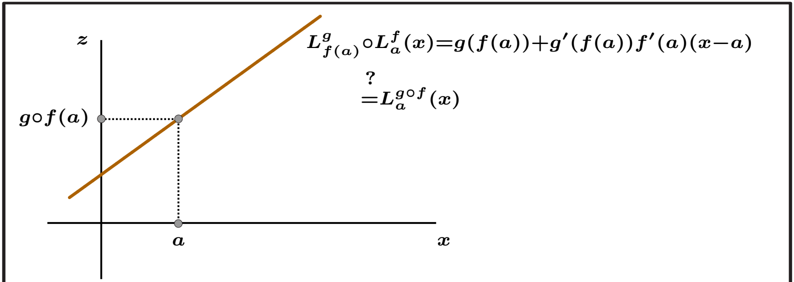
f(a)
y



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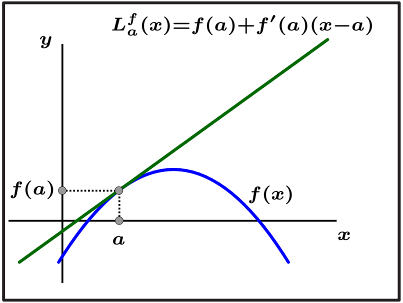

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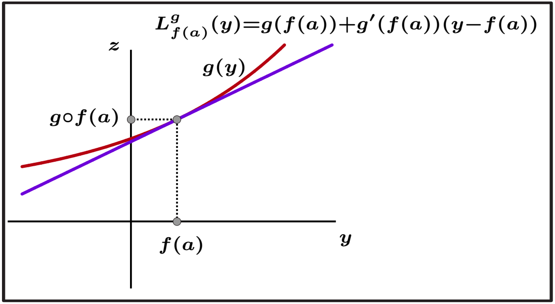

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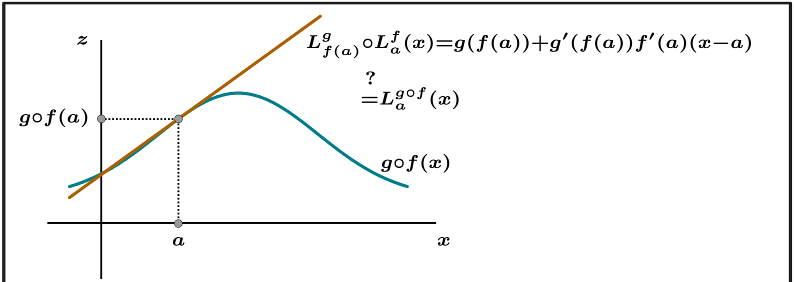

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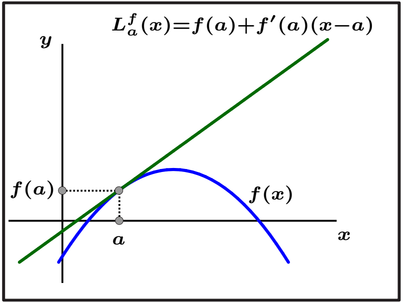

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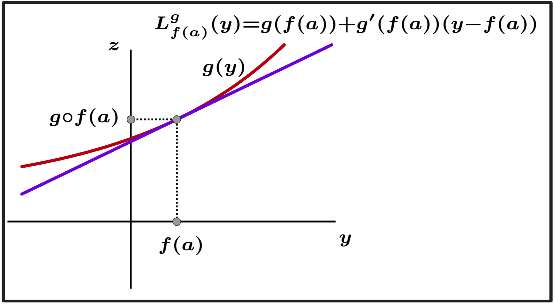

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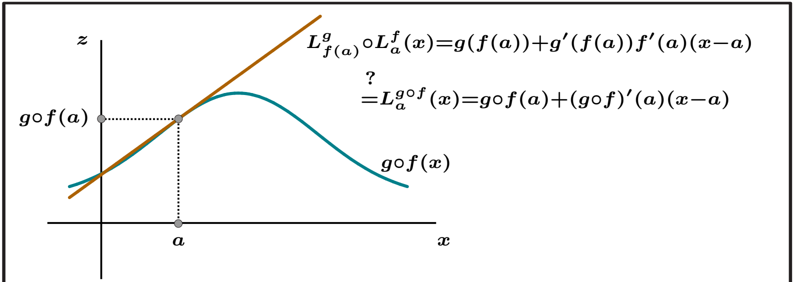

z
$L_{f(a)}^g \circ L_a^f(x)=g(f(a))+g'(f(a))f'(a)(x-a)$
$\overset{?}{=}L_a^{g\circ f}(x)=g\circ f(a)+(g\circ f)'(a)(x-a)$
$g\circ f(a)$
$g\circ f(x)$
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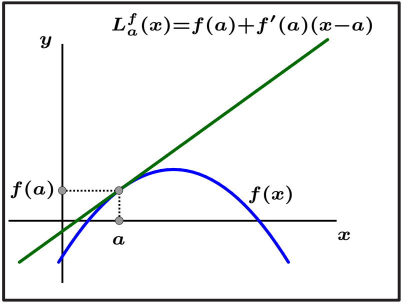
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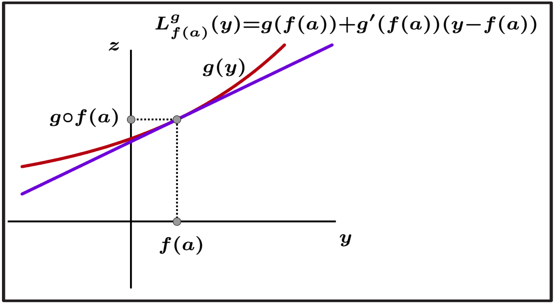
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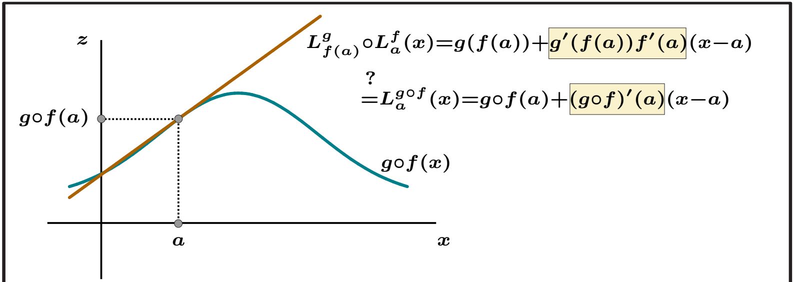
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z
$g \circ f(a)$
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Chain Rule

Theorem: [Chain Rule]

Assume that $f(x)$ is differentiable at $x = a$ and $g(y)$ is differentiable at $y = f(a)$. Then $h(x) = g \circ f(x) = g(f(x))$ is differentiable at $x = a$ and

$$h'(a) = g'(f(a))f'(a).$$

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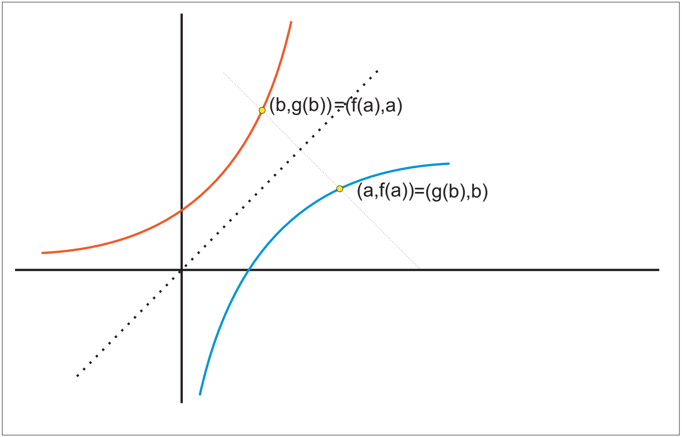
In particular,

$$L_a^h(x) = L_{f(a)}^g \circ L_a^f(x).$$

Inverse Function Theorem:



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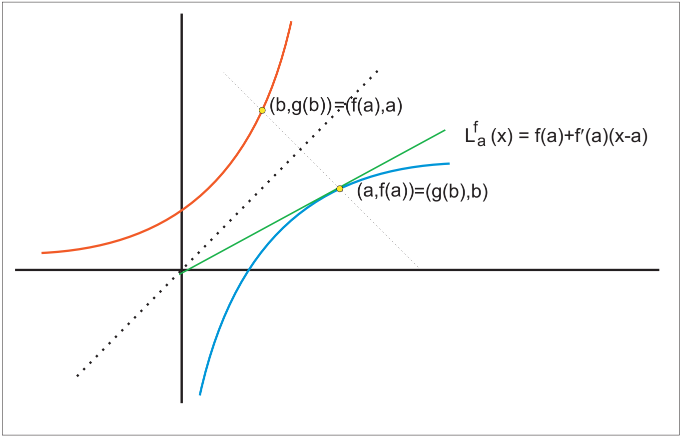

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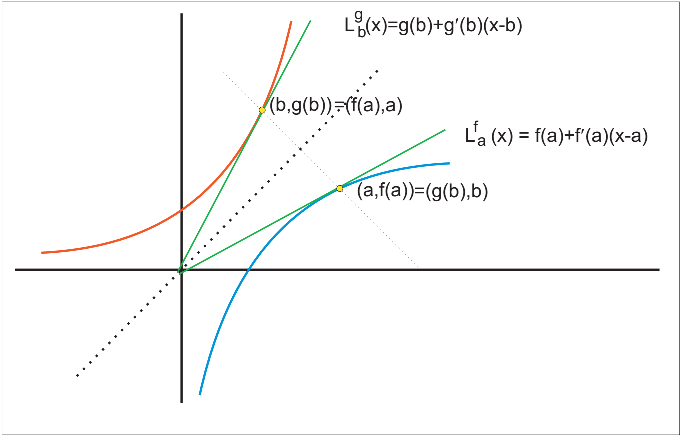

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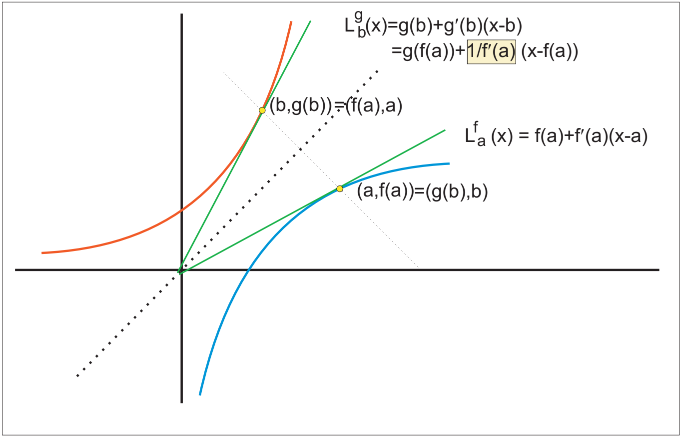

Inverse Function Theorem:

Theorem: [Inverse Function Theorem] (IFT)
Suppose that $f : (a, b) \to \mathbb{R}$, where $a < b$, and that $f(x)$ is one-to-one on (a, b). Let $g : f((a, b)) \to (a, b)$ be the inverse function of $f(x)$. If $f(x)$ differentiable on (a, b) with $f'(x) \neq 0$ for all $x \in (a, b)$, then $g(y)$ is differentiable on $f(a, b)$.
Moreover, in this case, if $x_0 \in (a, b)$ and $y_0 = f(x_0)$, then

$$g'(y_0) = \frac{1}{f'(x_0)}.$$

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Note : Linear Approximation can also be used to explain L'Hôpital's Rule.

Course Material:

https://www.math.uwaterloo.ca/ bafor-
res/UCM137/Lectures/BarbsM137Lectures.html

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