MAT137 and MAT139 at UTM

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What is this course?

MAT137 (now MAT137 and MAT139) is “calculus with proofs.”

It’s not analysis:

- We don’t construct the reals.
- We don’t talk about sequences of functions.
- We don’t prove the IVT*.
- We don’t get into too much depth with the theory of Taylor series.

It’s also not your usual “Calculus I/II”:

- We talk a lot about logic and proofs.
- We introduce the $\varepsilon$-$\delta$ definition of a limit and use it often.
- We do important proofs with the MVT.
- We work through all the details of (mostly Darboux) integrability.
- Very little “straight” computation on assignments.
Some background

We stand on the shoulders of giants.

The current form of this course, along with much of the design and all of the primary source material, were developed by the late Alfonso Gracia-Saz, at UTSG starting around 2013-14. At UTM, Jaimal Thind has shaped the materials of the course as well.

My experience with the course.

- Took the course as a student in 2005/6. I didn’t do very well!
- TAed the course in 2013/14.
- Instructed it at UTSG from 2014/15 through 2019/20.
- TAed the course at UTM in 2020/21 (it was a weird time).
  - UTM uses the same basic structure/primary source materials.
- Instructed it at UTM since 2021/22.
- Coordinating it myself for the first time this academic year.
Active learning in MAT137/9

History:

- Regular “lecture-style” until 2014/15.
- Flipped classroom for two weeks in winter 2016, as an experiment.
  - First math course at UTSG to try this!
- Flipped classroom for four weeks in 2016/17.
- Fully flipped the whole course starting 2017/18, with a comprehensive library of videos serving as the primary source material.
- Incorporated in-class polling (via MathMatize) at UTM in 2022/3.
The course selects for your favourite kind of students

Our students are not the math-competition-doing, advanced-topic self-researching, professor-correcting type.

Our students would probably describe themselves as “good at math in high school”.

But most of them have only taken math courses that encouraged memorization of formulas and algorithms, and plugging numbers into them.

(Or if they didn’t encourage them explicitly, the students optimized for their high-school courses with these strategies.)

That means they can usually handle “the details” of computations easily and don’t make many pre-calc errors.

They are quite capable of learning rigor, but they have usually never been exposed to it before.

These students are the ones we can “convert” into math people.
The sorts of things we expect students to prove on a test

Example 1

Let $a \in \mathbb{R}$. Let $f$ be defined at least on an interval centered at $a$, except possibly at $a$ itself. Prove that

$$\lim_{x \to a} f(x) = \infty \implies \lim_{x \to a} \frac{1}{f(x)} = 0.$$ 

Example 2

Let $a < b$. Let $f$ be differentiable with domain $(a, b)$. Prove that if $\forall x \in (a, b), f'(x) > 0$, then $f$ is increasing on $(a, b)$.

Example 3

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Prove that if

$$\lim_{x \to \infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = \infty,$$

then $f$ is surjective.
Problem solving

These are the tools that other departments want our students to have. We want students to be comfortable attacking new problems they have not seen before.

They should be able to:

- Make conjectures
- Adjust old methods to new situations (and see when they need to do that).
- Know when they’re right about something.
- What does it mean to prove something?
- Be comfortable struggling with a problem.

Whenever possible we avoid asking students to memorize things.
We start with logic

When I took this course in 2005/6, we had one week on basic pre-calc, then went right into the $\varepsilon$-$\delta$ definition of a limit, where everyone was promptly and completely lost.

Now we start with two weeks on logic, where we discuss:

- What are quantifiers?
- What are conditional statements?
- What does it mean to define something?
- What does it mean to prove something?
- What’s the basic structure of a proof, given the logical structure of the statement being proved?
Teach them what **is not** a proof...

"Thm": $\sqrt{xy} \leq \frac{x+y}{2}$

"Pf.": $xy \leq \left( \frac{x+y}{2} \right)^2$

$xy \leq \frac{x^2 + 2xy + y^2}{4}$

no explanations!

$4xy \leq x^2 + 2xy + y^2$

$0 \leq x^2 - 2xy + y^2 = (x-y)^2$

What are $x, y$?

(Ex: $x=y=-1$)

I start assuming what I want to prove
Thm: Let $x, y > 0$. Then $\sqrt{xy} \leq \frac{x+y}{2}$.

Pf. Since a square is always non-negative:

\[ 0 \leq (x-y)^2 = x^2 - 2xy + y^2 \]

\[ 4xy \leq x^2 + 2xy + y^2 \]

\[ xy \leq \frac{x^2 + 2xy + y^2}{4} = \left( \frac{x+y}{2} \right)^2 \]

Since both sides are non-negative:

\[ \sqrt{xy} \leq \sqrt{\left( \frac{x+y}{2} \right)^2} = \left| \frac{x+y}{2} \right| = \frac{x+y}{2} \text{ because } x, y > 0. \]
My students are sick of me talking about “proof structure”

Suppose you’re trying prove a statement like, hypothetically:

\[ \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in \mathbb{R}, \ 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon. \]

The “skeleton” of the proof is dictated by the statement itself:

Any proof of a statement like that looks like that

Fix an arbitrary positive real number \( \varepsilon \).

Let \( \delta = \) ________ (to be filled in later).

Fix an arbitrary real number \( x \).

Assume \( 0 < |x - a| < \delta \).

(Do work to prove that \( |f(x) - L| < \varepsilon \) under that assumption.)
Tell them what you expect of them

To whatever extent possible, give them grading schemes for assignment questions ahead of time.

I try to tell them explicitly:

- 2 points out of 5 for this question are just for getting proof structure right. You don’t need to have any thoughts to get these points!
- 1 point is just for everything being well-written, well-organized, and clear.

In general, I try to practice *radical transparency*. Tell students why you’re doing things and why they’re doing things.
Example in-class question

Let

\[ H = \{ \text{humans that have ever lived} \} \]

True or False?

1. \( \forall x \in H, \exists y \in H \) such that \( y \) gave birth to \( x \)

2. \( \exists y \in H \) such that \( \forall x \in H, y \) gave birth to \( x \)

Follow-up: We agree that Statement 1 is true.

Prove that Statement 1 implies every human has a grandmother.
We all know what even and odd numbers are. Complete the formal definitions of these concepts below.

**Definition**

Let $x \in \mathbb{Z}$. We say that $x$ is odd when

- $\forall n \in \mathbb{Z}, x = 2n + 1$
- $\exists n \in \mathbb{Z}$, such that $x = 2n + 1$

Every time I’ve asked this question (five or six times) the class is almost perfectly split in half.

**Follow-up: Prove this theorem**

The sum of two odd numbers is even.

Then show them some examples of ways *not* to prove this.
What is wrong with this proof?

<table>
<thead>
<tr>
<th>Theorem</th>
<th>The sum of two odd numbers is even.</th>
</tr>
</thead>
<tbody>
<tr>
<td>“Proof.”</td>
<td>3 is odd.</td>
</tr>
<tr>
<td></td>
<td>5 is odd.</td>
</tr>
<tr>
<td></td>
<td>3+5 = 8 is even.</td>
</tr>
</tbody>
</table>
Poll: What is wrong with this proof (part 2)?

**Theorem**
The sum of two odd numbers is even.

**“Proof.”**
We learned in elementary school that:

- even + even = even
- even + odd = odd
- odd + even = odd
- odd + odd = even.

So the sum of two odd numbers is always even.
Theorem
The sum of two odd numbers is even.

“Proof.”
\[ x = 2a + 1 \text{ odd} \]
\[ y = 2b + 1 \text{ odd} \]
\[ x + y = 2n \text{ even} \]
\[ 2a + 1 + 2b + 1 = 2n \]
\[ 2a + 2b + 2 = 2n \]
\[ a + b + 1 = n \]
Example in-class problem (lead-up to $\varepsilon$-$\delta$)

1. Find **one** value of $\delta > 0$ such that

$$|x - 3| < \delta \implies |5x - 15| < 1.$$ 

2. Find **all** values of $\delta > 0$ such that

$$|x - 3| < \delta \implies |5x - 15| < 1.$$ 

3. Find a value of $\delta > 0$ such that

$$|x - 3| < \delta \implies |5x - 15| < 0.1.$$ 

4. Find a value of $\delta > 0$ such that

$$|x - 3| < \delta \implies |5x - 15| < 0.01.$$ 

5. Let $\varepsilon > 0$ be an arbitrary, fixed positive number. Find a value of $\delta > 0$ such that

$$|x - 3| < \delta \implies |5x - 15| < \varepsilon.$$
From the assignment about inverse functions

MAT137 students often ask why we specifically choose \([0, \pi]\) as the interval to define \(\text{arccos}\). In this exercise, you’ll explore that question a lot.

(Split up into multiple parts and scaffolded.)

a. Find all intervals on which \(g(x) = \cos x\) is injective that cannot be extended to larger intervals on which that’s true.

b. Come up with a way of indexing them as \(D_\alpha\)’s that makes sense.

c. For each \(\alpha\), let \(g_\alpha\) be the restriction of \(g\) to \(D_\alpha\).

   Sketch the graphs of \(g_\alpha\) and \(g_\alpha^{-1}\).

d. For each \(\alpha\), find a formula for \((g_\alpha^{-1})'(x)\).

e. For each \(\alpha\), compute \(g_\alpha^{-1}(g(\pi/2))\) and \(g_\alpha^{-1}(g(137))\).

Congratulations, you’re now one of the world’s experts on cosine.
This one had several parts of scaffolding: it’s the Alternating Series Test!

From the assignment on sequences

Let \( \{ a_n \}_{n=1}^{\infty} \) be a sequence with the following properties.

- \( \{ a_n \} \) is \textit{wobbly}.
- Its \textit{buddy sequence} \( \{ A_n \}_{n=1}^{\infty} \) converges to 0.
- \( \{ |A_n| \}_{n=1}^{\infty} \) is decreasing.
- \( A_1 > 0 \) (this is just to cut out a parallel case that would work basically the same way).

Your task is to prove that \( \{ a_n \} \) converges.

Defined earlier on the same assignment:

- \textit{“buddy sequence”}: \( A_n := a_{n+1} - a_n \)
- \( \{ b_n \} \) is \textit{“jumpy”}: \( \forall n, \ b_{n+1}b_n < 0 \)
- \textit{“wobbly”}: \( \{ a_n \} \) is wobbly if its buddy is jumpy.
Consider the family of improper integrals of the form

\[ \int_0^\infty \frac{x^\beta \ln x}{(1 + x^2)^\alpha} \, dx, \]

where \( \alpha \) and \( \beta \) are real numbers. For which values of \( \alpha \) and \( \beta \) is the improper integral convergent?

State your answer in the form of a relationship between \( \alpha \) and \( \beta \) at the top of your solution, then include your justification below.

**Organize your work. You probably shouldn’t submit your first draft of this.**
The definition

\( \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in \mathbb{R}, 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon \).

is kind of like a (Python) class.

- The statement \( \lim_{x \to 4} f(x) = 7 \) is an instance of the class.
- \( \varepsilon \) and \( \delta \) are internal to the class/instance, and they don’t mean anything outside it.
- If you have an instance, it has a class method, which takes in \( \varepsilon \)’s and gives you \( \delta \)’s that do the thing.
- If you don’t have an instance yet, you have to write that class method.

This has been my most successful idea for explaining the scope of variables, and the difference between using a definition you know is true and proving a definition is true.
The expression

\[ \sum_{i=1}^{7} a_i \]

essentially executes the following pseudocode:

```
sum = 0
FOR i = 1 to 7
    sum = sum + ai
    i = i + 1
RETURN sum
```

It equals \( a_1 + a_2 + a_3 + \cdots + a_7 \).