FYMSiC Newsletter

A Calculus Vignette

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Tripatjit (enthusiastically): Jonny and I were studying together and we came up with an idea, but we were not sure if it made much sense. . .

Jonny (quietly): We were talking about the continuity and how a composition of two continuous functions is continuous.

Instructor (happily): I am curious to hear what two of my best calculus students have come up with.

Jonny (slowly, writing on his tablet): Well, we mentioned in the class that if f and g are two functions such that f is continuous at the number a in its domain and if g is continuous at the number $f(a)$ then their composition $g \circ f$ is continuous at a. Right?

Tripatjit (supportively, opening the textbook and pointing to a theorem): Look, our textbook says the exactly same thing.

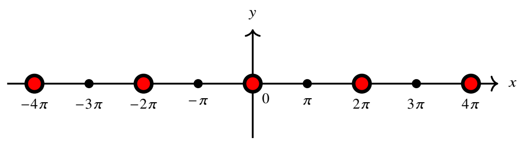
Instructor (cautiously): Yes, that's right. . .

Jonny (confidently): We wanted to check if this is true if the outside function is continuous on a closed interval. Like, functions $g : [\alpha, \beta] \to \mathbb{R}$ and $f : \mathbb{R} \to [\alpha, \beta]$ are continuous functions in their domains and for some real number a we have $f(a) = \alpha$. Is the function $g \circ f$ continuous at a?

Tripatjit (mischievously): If we do what we learned in the class we get $\lim_{x \to a} f(x) = f(a) = \alpha$ and $g(f(a)) = g(\alpha) = g\left(\lim_{x \to a} f(x)\right) = \lim_{x \to a}(g \circ f)(x)$. It is continuous!

Instructor (laughingly): Good try, Tripatjit! I know that you know that we have to be more careful with those limits because g is defined only for numbers that are greater than or equal to α. If we talk about $g(f(x))$ we have to make sure that x is such that $f(x)$ is in the domain of g.

Jonny (approvingly): Yes, we learned that strange things may happen. Tripatjit came up with this example: $F(x) = \sqrt{\cos x - 1}$. The domain of this function is the set $\{0, \pm 2\pi, \pm 4\pi, \ldots\}$ and its graph looks like this:



Tripatjit (cautiously): This is why we wanted to talk to you. If we take $f(x) = \cos x - 1$, this function is continuous everywhere. The function $g(x) = \sqrt{x}$ is continuous in its domain $[0, \infty)$. Clearly, $F(x) = (g \circ f)(x)$ and, for any integer k, $F(2k\pi) = g(f(2k\pi)) = g(0) = 0$. Since f is continuous at $2k\pi$ and g is continuous at $f(2k\pi) = 0$, we think that it would be nice if we can say that F, as the composition of two continuous functions, is continuous at $2k\pi$.

Jonny (thoughtfully): We have checked a few calculus textbooks and all of them discuss the continuity of functions defined on an interval or the union of intervals. Does it even make sense to talk about the continuity of functions that are defined on a finite or a countable set?

Instructor (knowledgeably): It does, but we need more general settings. Maybe you know, maybe you don't, there is a branch of mathematics called *topology*. . .

Reference

Burazin, A., Jungić, V., & Lovrić, M. (2024). A detailed look at continuity in Calculus textbooks. *International Journal of Mathematical Education in Science and Technology*, 1–17. Retrieved from: `https://doi.org/10.1080/0020739X.2024.2337943`