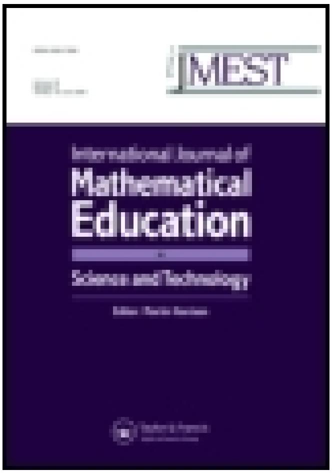

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In plane view: an exercise in visualization

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In plane view: an exercise in visualization

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We offer a problem in visualization which uses multivariable calculus concepts. The problem is essentially to describe (mathematically) what we can see on one mountain while sitting on an adjacent mountain. We present how our students, working in groups, attack the problem and the issues which surround the solution strategies produced. We have successfully used this problem for a number of years in our courses, devoting several class periods and about 2–3 weeks outside of class to student development of a solution strategy. The problem serves to develop visualization skills, verbalization of mathematical concepts, and implementation of problem-solving notions in mathematics including gradients, optimization, integration, surface area, and programming.

1. Introduction

For the past five years we taught in a rich environment as part of a team of faculty teaching in an Integrated First-Year Curriculum in Science, Engineering, and Mathematics (IFYCSEM). In IFYCSEM the technical courses (calculus, mechanics, electricity and magnetism, chemistry, statics, computer programming, graphics, and design) are all put together in three 12 credit quarter courses [1]. This course is team taught to a cohort of 90 entering students by a group of eight faculty. The faculty team for 1994–95 consisted of two from mathematics, one from physics, two from chemistry, one from electrical engineering, one from computer science, and one from mechanical engineering. Teaching in this environment gives an opportunity to relate material and a degree of freedom not offered in a traditional calculus course. Opportunity and freedom come from team teaching and access to laboratory time for mathematics, coupled with a knowledge of what students have seen and are doing in other areas, physics for example. This paper deals specifically with linking programming, visualization and mathematics.

Most of the contact hours in this class were conducted in a computer lab of 30 NeXT workstations, each equipped with *Mathematica*. Students have access to this technology for *all* their work—class, projects, homework, lab write-ups and exams. In IFYCSEM we emphasize cooperative learning and almost all class meetings have a strong component of group work and many assignments are for groups, perhaps with one submission from the group.

In IFYCSEM we used *Mathematica* early in the course, both as a principle tool for mathematics and as the introductory programming environment. Thus we are

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able to build on this expertise in a number of projects and problems throughout the course. In this paper we describe an application of mathematics and a use of *Mathematica* in visualization. Visualization is emerging as an important aspect of teaching and learning mathematics [2], especially in light of using technology for graphing and spatial manipulation.

2. The visualization problem

We introduce the following problem in class. We usually begin the problem in a room in which there are no computers so the students will do a bit of visualization of their own without turning to computers to 'crunch' functions and numbers.

For the function

$$f(x,y) = \frac{(x^3 - 3x + 4)}{(x^4 + 5y^4 + 20)}$$

suppose your eye is precisely on the surface $z = f(x,y)$ (see Figure 1) at the point $(2{\cdot}8, 0{\cdot}5, f(2{\cdot}8, 0{\cdot}5))$. You look to the left, i.e. in the direction (roughly) $(-1, 0, 0)$. You see a mountain before you.

(a) Determine the point on the mountain which you can see which is nearest to you.
(b) Describe as best you can the points on the mountain which you can see from the point $(2{\cdot}8, 0{\cdot}5, f(2{\cdot}8, 0{\cdot}5))$.
(c) Determine the amount of surface area on the mountain which you can see from the point $(2{\cdot}8, 0{\cdot}5, f[2{\cdot}8, 0{\cdot}5])$.

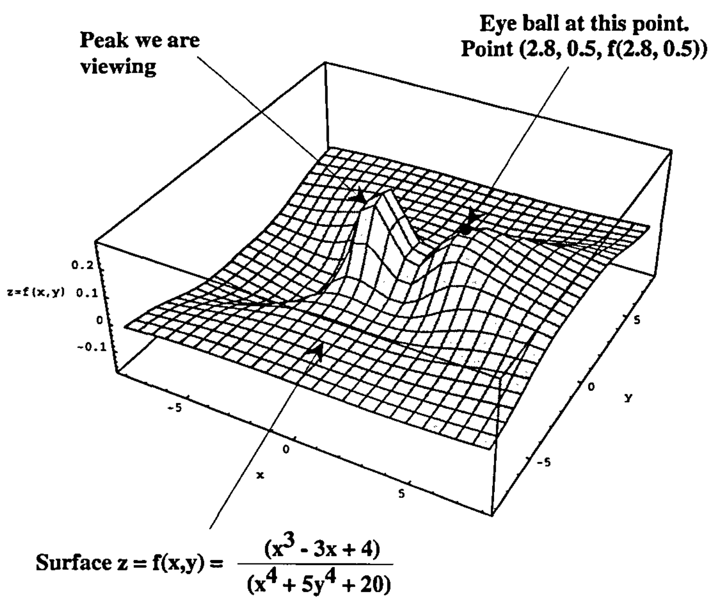

Figure 1. Surface $z = f(x,y) = (x^3 - 3x + 4)/(x^4 + 5y^4 + 20)$.

That is it. We give the students the statement of the problem with the figure and say, 'Go to it!'. Since they are always working in groups they turn to their neighbour and start buzzing. We circulate around the class, listening in on group discussion and responding to individual queries. From time to time, we summarize group progress for the class and ask groups to make progress reports during class time.

3. Some background observations

We spend three days of class time on this problem—usually over a two-week period. One year we spent four consecutive class periods on the problem. We require students to submit individual write-ups two and one-half weeks after the problem has been introduced in class. While students work in small groups both in and out of class each student is responsible for writing up an individual report. At the one and one-half week mark we require individuals to submit a one page progress report indicating who they are working with, what ideas are emerging, and what progress they are making.

When we first assigned this problem several years ago we did not assign part (c), nor did we really have an in-hand solution to (a) or (b). We told the students this and some were quite concerned that we would assign a problem to which we did not know the answer. 'Life is like that in the real world', we say, i.e. your boss will not know the answer to a question she asks you, but she will have some idea as to whether or not it is a reasonable question and if you can contribute to its resolution. Moreover, she will be there to assist you, advise you, respond to your ideas, and to offer tools, equipment, and other resources should you need them. 'Welcome to the real world!'

Not only did we not have a solid approach to the problem before going to the students with it, but we did not have the many rich ideas and varied approaches which have emerged from students' approaches over the several times we have assigned this problem! We believe these approaches illustrate the breadth of student initiative and creativity and we share them with the reader in the hopes that the reader will try this problem or some variation with students.

Some samples from progress reports indicate the directness of students in sharing their feelings and observations:

> Initially in the class I had no idea of how to go about solving this problem. The problem seemed beyond my knowledge. With a little discussion among my friends in the class and help outside of class I now believe that I actually may be able to understand what is going on and solve the problems.

> When I first saw the project it looked nearly impossible. That view changed rapidly as I began to chat with my colleagues. The ideas began to flow and the problem quickly became manageable.

> I have no huge progress to report on this subject, but I do have several ideas.

> The first idea that we came up with is the idea that if we could make the second hillside, the one we see, a line in two space, a line perpendicular to that line running through our position point, would be the shortest distance. The

problem with this is that one, we are in three space not two, and two the perpendicular lines through the hill and our point may not be actually visible to us. Even though this idea was not usable it led to other good ideas.

When this problem was first assigned, I approached it with some trepidation. It is scary when you are faced with a problem that even your professor has never solved. After talking with fellow students, I have seen that this problem will not be as evil as it first seemed.

The best title on the final report was 'Making Molehills Out of Mountains'. The best observation by a student was, 'These failed ideas often led to better ideas in the long run and were usually not wasted time'.

4. What do students do?

In general, the class does a wonderful job on this problem. Indeed, they do a good job on most problems in which cooperative learning is used for no one student is forced to battle a tough idea alone. There is always another point of view sitting in the next seat with a colleague. Individuals and small groups do some very unique things with the problem. We have outlined these approaches below. There were a good many different approaches and while some consensus approaches emerged there were even different variations on these.

We make some general observations to the students after the mid-problem progress report has been submitted. We do this through email to the class. We use email to communicate on a routine basis in IFYCSEM.

- You should restate the problem or at least not just jump into the problem without some introduction and identification of the issues and the function.
- Be careful to define your terms and be sure you have uniqueness, e.g., do not just say 'the plane' say 'the tangent plane at the point (x, y, z)'.
- Identify and give credit to those with whom you work.

In the discussions of the class the notion of not seeing 'below' the tangent plane at the 'eye-point' emerges quickly and then issues such as determining where the tangent plane intersects the surface follow. While working with a group it becomes apparent that not all approaches are linear, i.e. (a) then (b) then (c) etc. Students jump about in the problem, asking questions which sometime address the part they think they are considering, but often lead to looking at issues related to another part of the problem. Indeed, in one section a student immediately offered to determine the (x, y) region of the viewed mountain cut off by the tangent plane and use *Mathematica*'s `ImplicitPlot` to plot the relation `TangentPlane[x,y]-f[x,y]==0`. Then plot this implicit plot over the contour plot of the function $z = f(x, y)$.

We now address the main themes of the problem and offer student approaches.

5. Find the closest visible point

Students offer a number of different ideas and approaches in trying to solve (a) Determine the point on the mountain which you can see which is nearest to you.

A number of students argue that the points of intersection of the tangent plane at the eye-point and the surface must contain the 'nearest point' since the

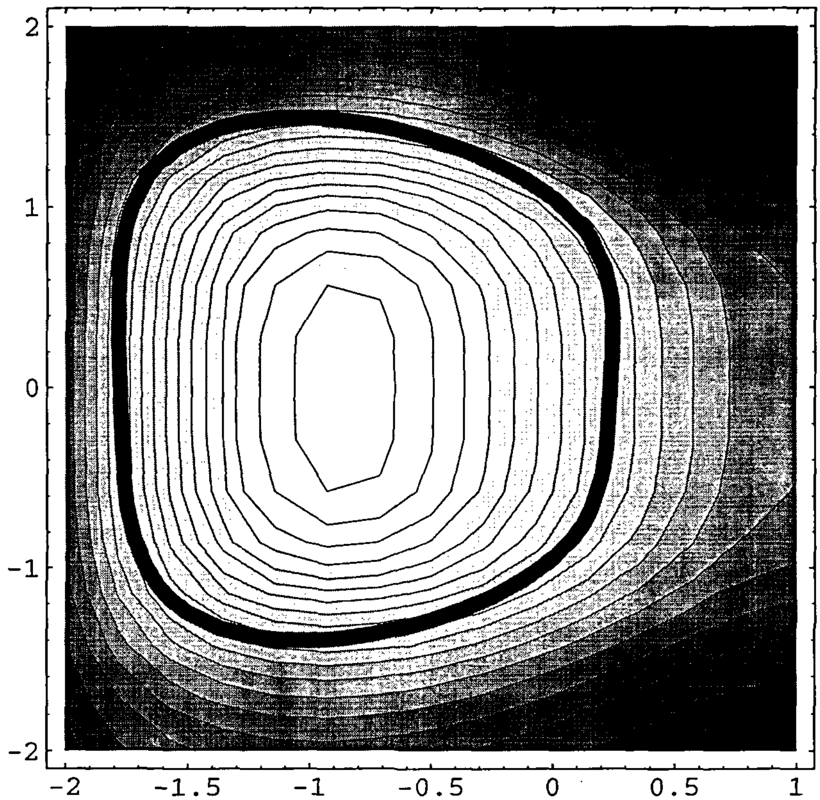

Figure 2. Merged plot of points of intersection of the tangent plane at the eye-point with the mountain being viewed and a contour plot of the function $z = f(x, y)$.

mountain slopes away above this tangent plane. But it is not clear that this is so and several students offer 'hand-waving' sketches to counter this notion. Indeed, later they find out the tangent plane does contain the nearest point, namely the eye-point itself(!) and some have a tough time getting their algorithm to not converge back to the eye-point.

Using a simple loop, actually double loop, one student offered a mesh of points which provide the x and y limits of a reasonable region of visibility and then used an even finer mesh inside this one (say 0·005) and tested all points inside the boundary to find the closest point to be (0·241, 0·47, 0·162) at a distance of 2·559 units.

Two bright students decided to first define the region of visibility (interior region in the (x, y)-plane of the projection onto the (x, y)-plane of the intersection of the tangent plane at the eye-point and the surface) by estimating a collection of points on the intersection through clicking on the above implicit plot in Figure 2. They proceeded to fit a polynomial curve, $x = g(y)$, to the sampled data points, knowing they could not solve explicitly for x or y. (*Mathematica* will permit the user to click on the figure and with a hairline obtain the coordinates of the point at which the cursor is located.) Then they minimized the distance function from the eye point to the surface, i.e. $d(x, y) = [(x - 2{\cdot}8)^2 + (f(x, y) - f(2{\cdot}8, 0{\cdot}5))^2]^{1/2}$. Here the $x = g(y)$ function fitted to the boundary data was substituted into

$d(x, y)$ making $(d(x, y) = d(g(y), y)$ a function of one variable which was easy to minimize. Again this presumes the nearest point occurs at the intersection of the surface and the tangent plane. One of these students then went on to state it best when he said, 'A better approach was decided as working backwards and find all the visible points on the second mountain. Then, use the `FindMinimum` command in *Mathematica* to find the shortest value of the distance function, $d(x, y)$, over this range'.

Another student, after determining the equation of the tangent plane, $p(x, y)$, defined the distance function

$$d(x, y) = \text{If}\,[f(x, y) > p(x, y), ((x - 2{\cdot}8)^2 + (f(x, y) - f(2{\cdot}8, 0{\cdot}5))^2)^{1/2}, 15],$$

the latter to keep the distance high. He then applied *Mathematica*'s `FindMinimum` command to $d(x, y)$.

A weaker, but persevering student, found $f(x, y)$-tangentplane (x, y) and went through various y values 0·41 and 0·425 in increments of 0·001 (after seeing the contour plot of the slice on the mountain) printing out the x value of the intersection and the distance to the eye point in one table. Then he selected the point with the smallest distance (0·244, 0·417, 0·162 86) which was 2·5573 units away from eye point.

Several students suggested that the nearest point on the mountain to our eye would be one for which a line from the eye to the point is perpendicular to the surface—but no justification or apparent use of this idea was pursued.

All seemed to be getting something like (0·244, 0·4, 0·163) as the nearest point on the opposite mountain with a distance from eye-point to nearest point of 2·558 units.

6. Describe what you can see on the other mountain

In attempting to describe what you can see on the other mountain, most students went with determining a boundary of region you could see. This meant determining the points of intersection of the surface with the tangent plane. These points were not on a closed form relation or function, but consisted of a number of points obtained by numerically solving for y coordinates when an x coordinate was offered.

As to what one could see at the 'top' of the mountain. There were a number of ideas, including an initial interest in finding the highest point on the other peak. To this approach a number of students immediately said, 'But you probably can't see that top point'.

Students would attempt to get a window on what we can see by swinging out from the centre until the surface is no longer 'hit' by our line of vision, the same would apply to swinging up to get a top view limit. But, in practice, these ideas were never implemented.

One idea that did work was, in their own words, 'Place a stick end where your eye is and let it fall until it lays on the surface of the viewed mountain. You cannot see beyond that point of contact in that line of view'. This proved to be a very powerful image for students and it led to the following idea:

The line (L) from our eye-point to the other mountain of the surface which just

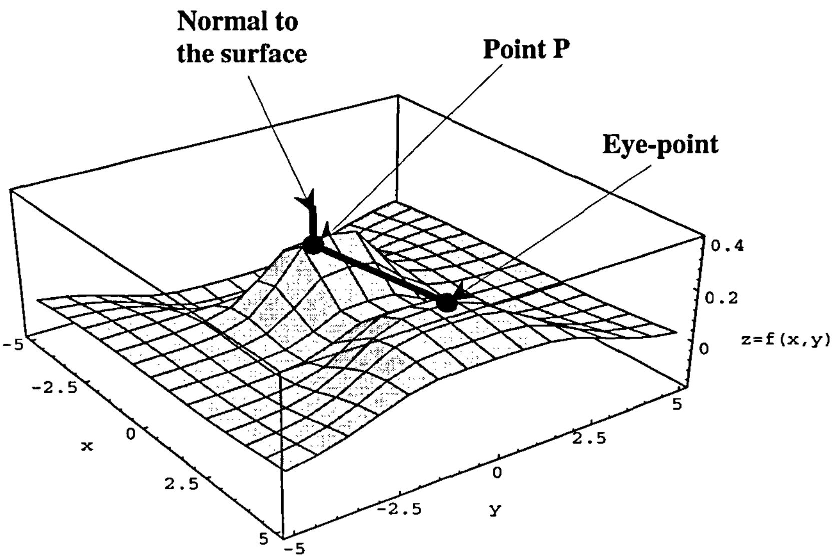

Figure 3. Plot showing the line connecting eye-point to arbitrary, but visible point, *P*, on second mountain and the normal to the point *P*.

lies on the other surface will touch the other surface at a point at which the normal (N) to the surface at that point and L are perpendicular, i.e. the dot products of the directions of N and of L must be 0 (see Figure 3).

or

A point on the second mountain is visible if the dot product with the vector from the eye to this point, L, with the normal vector to the surface at this point, N, is greater than zero, i.e. the angle between the vectors is less than 90°. Here it is noted that the z value of the points in question on the surface have to be greater than the z value of the point with the same (x, y) coordinates on the tangent plane at the eye-point, i.e. the point is visible if it is also above this plane.

Moreover, one student made an array of all visible points over a fine mesh within such defined boundaries and then checked the distance for each of these visible points with a nice `Do` loop looking for minimums.

Another clever idea was to define the visible point function, $v(x, y)$. After determining the equation of the tangent plane, $p(x, y)$, define the visible point function $v(x, y) = \text{If}\,[f(x, y) > p(x, y), f(x, y), f(2{\cdot}8, 0{\cdot}5)]$

6.1. *Ideas on tangent plane boundary*

The following were some of the ideas to attempt to characterize the boundary created at the intersection of the tangent plane to the eye point and the opposite surface:

1. `ImplicitPlot` intersection of tangent plane at eye-point and the surface.
2. Numerically solve the intersection of tangent plane and surface.

3. Fit a curve through the numerical set of points to outline the intersection of tangent plane and surface.

One student stated, '... that the parametric representation of the tangent plane was less useful than the Cartesian one'.

6.2. *Ideas on top ridge boundary*

In attempts to determine the points along the top of the opposite mountain which serve as the boundary between those points we can see on the 'front' and those we cannot see on the 'top' or 'back' students came up with the following approach.

1. Consider points $(x, y, f(x, y))$ on the surface which are (a) on tangent plane and (b) where the dot product of normal to surface and vector connecting that point and the eye point is zero—see Figure 3.
2. Take the above points (their projection into the x–y plane) from (1) and fit an eighth (or degree of choice) degree polynomial to them and use this as part of the x–y boundary of visible points.

7. Determine the amount of surface area on the mountain you can see

A number of very different approaches emerged and we list them here.

1. Visualize two cones—one using the smaller side (due to the tilt) of the mountain and the other using the larger side. Obtain the average of these surface areas and then determine the angle out of the full 360 degrees which is subtended by the extreme (left and right) tangent lines from the eye point.
2. Offer a mesh of points which provide the x and y limits of the region of visibility and then integrate the element of surface area over each square in the mesh to get approximate surface area of 2·232 square units.
3. Confine the region of visibility in the x–y plane by approximations, e.g. by parts of circle and a line and then integrate the surface area.
4. After fitting polynomial curves to data of the ridge points (found in previous section) and the intersection of the tangent plane with the opposite mountain the surface area element is integrated over the region as a double integral.
4. Approximate the region over which we should integrate by semicircle and two straight lines.
6. Write a loop to integrate for fixed x from the lowest visible point to the highest visible point using a mesh and a test to see if the element of surface area should be accumulated.
7. Slice vertically across the range and add the 'arc length times the change in x' (actually the little elements of surface area)—checking to be sure each point which would contribute was visible.
8. Estimate the average slope of the surface (did not say how) and over the mesh placed on the visible region multiply the area of the little elements of x–y squares by the slope. Then add up the surface area estimate for each little rectangle.
9. Break the underlying x–y region into triangular regions, but one needs to then integrate the elements of surface area to get ther estimate of surface area above these elements.

10. Create a simple function newf $(x, y) = \text{If}\ [f(x, y) > \text{plane}\ (x, y), f(x, y), -0{\cdot}2]$ to 'grab' all the points which were above the tangent plane and on the surface and use it in numerical estimates of surface area.

8. Conclusions

This project poses a real challenge to students. Students used calculus, gradients, geometry, tangent planes, equation solving, *Mathematica* programming, and numerical methods for determining points on the intersection of the plane and surface. `ImplicitPlot` proved helpful as well for visualization. In solving this problem students used known concepts in new contexts and in some cases discovered new concepts. They worked through a complex process to assemble a solution to the visual problem before them.

We highly recommend this project to enliven your class, to engage the students, to challenge visualization skills, and to give students an opportunity to work in teams to solve a complex problem.

Source and acknowledgement

A *Mathematica* notebook and an ASCII version of this problem, with solution and comments is available (under title OverView), as a part of a larger National Science Foundation project effort, 'Development Site for Complex, Technology-Based Problems in Calculus', NSF Grant DUE-9352849. This notebook along with other problem sources material developed with support of the grant is available on the World Wide Web under the address: http://www.rose-hulman.edu/Class/CalculusProbs. Web site preparation was done by Dr Aaron Klebanoff, a bright, young colleague at Rose-Hulman Institute of Technology.

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