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4. Conclusions

In this note, we have described examples from various fields in mathematics, where the same cognitive skills are required. This situation is not rare, and has been met in []. In [4] we described how the same flowchart can describe the solution of different problems. However, these topics are often met in different courses, and, generally, students do not notice the similarities between the fields.

The simplest way to have these similarities noted by the students is to ask teachers to point them out explicitly. Another way, better from our point of view, is to build 'integrated courses', where the same teacher is in charge of various fields, and the mathematical notions are not taught according to their specific field, but according to the mathematical thinking they require.

Finally we wish to make the following remark: replace the Euclidean plane (or 3-dimensional space) of the geometry by any vector space with an inner product, or by a metric space. The metric properties we described in section are still valid. For example, consider the space of square Riemann-integrable functions on a closed interval with the inner product defined by $< f, g >= \int_a^b f(t)g(t)\,\mathrm{d}t$. All the metric relations we saw in section have a translation into integral relations. The same phenomenon occurs, of course, for all the popular metric relations of a course in Linear Algebra, such as the Cauchy-Schwarz inequality, the triangular inequality, and so on (cf [5]).

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Getting to the mall: an activity for problem-solving

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The note offers a general discussion of the use of complex, technology-based problems and a description of such a problem that is useful in a number of classes in which optimization is a goal, but modelling is essential. The problem demands the use of technology and permits a number of attacks. The problem is essentially to describe (mathematically) the region surrounding a shopping mall neighbourhood after a high-speed highway is put in place near the mall. The note shows how students, working in groups, attack the problem and the issues which surround their solution strategies. This problem and other problems like it have been used successfully for a number of years in courses, devoting several class periods, spread out over 2–3 weeks, coupled with activities outside of class for student development of a solution strategy. These types of problems serve to develop visualization skills, verbalization of mathematical concepts, and implementation of problem-solving notions in mathematics including optimization, curve fitting, integration, and symmetry.

1. Introduction

A primary goal, perhaps for some *the* goal, in teaching mathematics courses is to enable students to practise problem-solving while learning a body of mathematics. This is done to motivate the mathematics and to produce useful 'knowers of mathematics.' Moreover, many students are truly intrigued with applying their mathematics and some, many in fact, find mathematics more palatable if there is something that interests them attached to the study at hand. Good problem solvers do not just develop skills to solve the problems at the end of each section or even the review section of the text. They develop a mindset that says, 'Bring on the problem. Let me see what I can do to solve it.' Good problem solvers know their tool kit and know how to use the tools in context, not just in abstract or textbook settings. They know the limitations of certain strategies and the opportunities for other strategies. Problem solvers learn by doing. They grow in problem formulation, solution, and interpretation skills by experiencing these on a regular basis. Accordingly, we give students complex problems to solve.

In addition to the need for students to deal with problem complexity there is the belief that students need to know how to use technology as part of their expanded problem-solving tool kit. Thus we encourage them to use computer technology to explore a problem setting and to assist them in approaching the issues of the problem, if not enabling a full and satisfying solution.

It is for these reasons that over years of teaching we have used complex, technology-based problems as motivation and as assessment activities in courses.

2. Assessment value of complex problems

Teachers can assess students' problem-solving skills by observing almost all aspects of the students' activities involved in solving these complex problems. Indeed, in assigning such problems we have a rich opportunity to assess 'student initiative, creativity, and discovery; flexibility and tolerance; communication, team, and group self-assessment skills; mathematical knowledge; implementation of established and newly discovered mathematical concepts; and translation from physical descriptions to mathematical models.' [1, p. 120] Listening is an invaluable assessment tool for the teacher while wandering about the room and responding to calls for help/clarification from student groups. One can see, first hand, students' abilities to build sophisticated solution strategies from simple parts, e.g. geometry principles and optimization of functions of one variable in the case of the problem discussed in this paper. A good assessment instrument to employ in this setting is a journal describing the process of each group as they move towards success. Students' writings offer descriptions of the process, including strategies which did not work as well as those that do work, and they show the state of the students' problem-solving strategies and how they move ahead. In simple, one-step problems with routine, algorithmic responses, one tests recall mostly, not problem-solving abilities. Only through complex problems can one really see the students' thinking out loud while working on a solution.

3. Criteria for good problems

What are we looking for in a good, complex, technology-based problem? We outline a set of criteria.

- The setting for the problem should engage the student.
- The situation should be interesting, possibly involving a real application.
- The nature of the vocabulary, notions, geometry, etc. should assure reasonable access at the start.
- There should not be an obvious attack. The front of the problem should be wide enough to permit several different approaches. Indeed, there ought to be a number of levels and approaches afforded by the problem's complexity such that there are multiple strategies possible.
- The problem cannot be a simple algorithmic notion and it should not be a routine end-of-chapter problem. There should be a level of complexity that offers more than just a one- or two-step algorithmic approach.
- The problem should serve multiple goals for the teacher, i.e. help clarify issues and tools by putting them into contexts and interesting settings for the students.
- There should be meaningful (not contrived) uses of technology. (Of course, some readers believe that good problems do not *always* involve technology, but when we are trying to emulate a real-world situation we find it wise to try to use technology in the problem setting.)
- There should be satisfying answers to the questions, 'Why is this problem worth considering? Is it worth my students' time?'

4. Time issues in assigning and evaluating this problem

Usually we give this and other problems like it over an extended period of time. Several visits in class are devoted to (1) introduction, (2) touching base to see how students are initially approaching the problem and to permit sharing of ideas and strategies among students, and (3) final overview so all are well equipped to go to their write-ups for final submission.

The total class time spent on the problem should be no more than that of a traditional period, except that one should spread this time over three different days. We always assign teams, usually teams of two for a problem of this level, although teams of three can work. However, with teams of three there may be some coasting which takes place and we would prefer that ALL students get deeply involved. Students produce a solid write-up with good mathematics, documented computer use, and defence of the method, coupled with a summary or overview of their strategy.

Assigning extended problems of the sort described here is worthwhile and it is in the spirit of making calculus a leaner and livelier experience [2]. Moreover, students' skills are enhanced when they are required to use these skills in new contexts and settings. Raising the bar, moving our expectations ever higher, as teachers, will raise students' attention and energies as well. If we lead them they will follow, so let us lead them to a higher level of thinking while using the fundamental skill set they possess through solving complex problems. This is a worthy and noble teaching goal.

Team write-ups are always better than individual write-ups, for quite often there is a self-correcting effect in place as one student writes a draft of a section and another team member does a critical read of the write-up. Thus, the papers read more smoothly. If one is comfortable assigning group grades, and there are a number of theories and practical approaches one can use, then groups of two cut the grading in half; three by one-third! More importantly, students know that in the world outside academe they will be working in groups and it will serve them well to practise interpersonal skills, delegation and receipt of responsibility, as well as constructive feedback.

5. The problem

We offer up a problem which meets our goals and which we believe any teacher can use in an appropriate setting. Where could one use this problem? We have used it successfully in calculus—single- or multi-variable, optimization, or mathematical modelling courses. We state the problem as we have posed it several times. Of course one could use different numbers, units, and scenarios, e.g. give the units as km/h instead of miles/h.

> The average driving speed to reach a shopping mall in a suburban area through unimproved roads is 30 miles/h. People seem to be willing to spend no more than one hour of driving time to reach the mall. Hence the 'neighbourhood' of the mall determined by this transportation constraint is a circular region centred at the mall and having a 30 mile radius.[1] Suppose a new east–west highway is built, passing ten miles due north of the shopping mall and that the driving speed on the highway is 55 miles/h. Determine the new 'neighbourhood' for the shopping mall in view of the option to take this route.

Usually this statement of the problem is all that we give the students. In fact often during the class, in preparation for the assignment, we simply draw a circle with the highway about one-third of a radius above the centre and parallel to the floor, we describe the situation, and we ask the students to run with it, either at the boards or at their desks, usually in teams of two or three.

However, we could offer a more guided presentation as presented in points (a)–(f) below. We suggest that giving such guidance takes away from a richer discussion of the problem formulation. For example, students will debate whether folks coming in on the highway should go along the highway to a point directly north of the origin (mall) and then drop south or whether there is some 'cut-off' they should take. This evolves into a question of optimality, i.e. what is the path one should take if one is on the highway to get to the mall in the quickest time? Using symmetry we can confine our discussion to the northeast corner of our region. For those who live outside the circle, how does one approach the highway? Does one drop south immediately and then travel west? OR does one cut 'cross-country' in a somewhat southwesterly direction towards the highway? And if so, at what angle? Are the angles for the cutoff points the same, i.e. first when to enter the highway and then when to leave the highway?

(a) Place the mall at $O = (0, 0)$ and place the new highway along the line

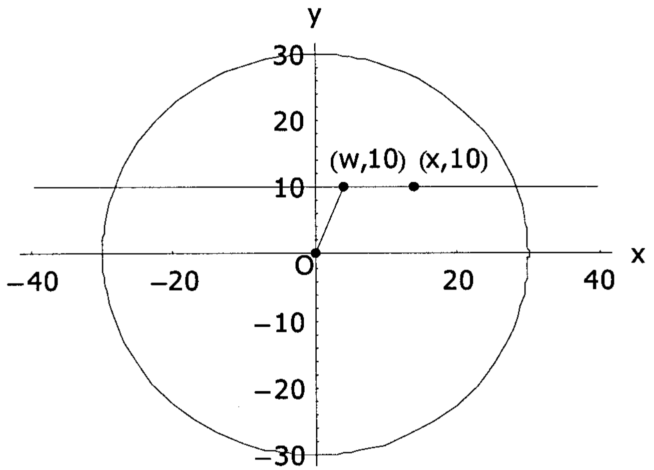

Figure 1. Helpful mall figure which is suitable for students to suggest different paths from an arbitrary point $(x, 10)$ to the Mall $= O$ in optimal time through a point $(w, 10)$.

$y = 10$ (see figure 1). Assuming a path that follows the highway from the starting point $(x, 10)$ to a 'cutoff point' $(w, 10)$ (where $x > w$) and then proceeds directly to the mall at $(0, 0)$, find the distance w that minimizes the total travel time from $(w, 10)$ to the mall at $(0, 0)$. Does w depend upon x?

(b) Using results from (a), find how far out along the highway a vehicle can be in order that travel time to the mall is one hour or less.

(c) Now consider people north of the highway and ascertain a path of least time they should take to get to the mall.

(d) Using the results from (c), determine a boundary north of the highway such that all who are within that boundary can get to the mall in one hour or less.

(e) What path should people take if they live within the 30-mile circle and south of the highway to get to the mall in the shortest time? If they live outside the 30 mile circle and south of the highway?

(f) Now offer a new neighbourhood for the shopping mall such that all people in that neighbourhood can get to the mall in one hour or less, using normal suburban roads and the highway.

6. Assumptions

It is very important that we address assumptions before formulating this problem. Students will make a number of them and most are needed. Some are superfluous. This is the time to recognize the tradeoff between tractability of the analysis and the reality of the situation. These assumptions will usually include:

- At all times people can drive as the crow flies, i.e. in any direction they need/ want.
- Drivers can enter and exit the highway anywhere along the highway.
- Drivers will travel the path of least time.
- No time is consumed in the switch from highway to 30 mile/h road and back.

- There is instantaneous acceleration.
- There is no slowing of a car due to other traffic.
- The two speeds remain constant in their domains.
- People south of the highway will go north to the highway and those north of the highway will go south to the highway.
- We set up an x–y axis system with the mall at the origin $(0, 0)$ and the highway running along the line $y = 10$.

7. Student actions and reactions

Now let us consider what we can expect students to do, to gain, and to produce. We shall use the sketch in figure 2 as our reference, but realize that constructing this sketch is part of the early struggle for the students. Indeed, the first thing students tend to do is consider how someone living on the highway gets to the mall both from within the one hour neighbourhood and from without. They are interested in this because they see right away that it would be worthwhile to determine just how far out on the highway people can be and still get to the mall in one hour. In their haste to address this 'furthest distance' issue they will often make the assumption that the driver simply drives until directly north of the mall (V) and then makes a sharp left turn (assuming the driver is coming in from the east), heading straight south to the mall. The goal then is to figure out how long it takes to get to the intersection of the highway and the circular boundary of the one hour neighbourhood. With the remaining time in the allotted hour, they can travel at 55 miles/h to the farthest point along the highway, U. Usually someone in the room suggests that perhaps the perpendicular route to V (see figure 2) and down to the mall may not be the best route in terms of minimizing time. So the point P is introduced.

The path to get from a typical point $(x, 10)$ (see figure 1) to the mall depends upon the choice of point $P = (w, 10)$ at which one cuts off the highway and enters the slower speed region. Students quickly see that they wish to minimize this time,

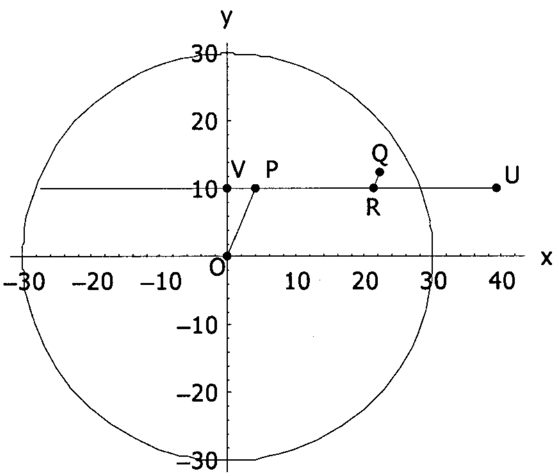

Figure 2. Mall figure with $O =$ Mall, $V =$ vertical entry point to highway, $P =$ entry point to highway, $Q =$ typical outlying point, $R =$ optimal entry point to highway, $U =$ furthest point reachable (using an optimal path) along the highway in one hour.

i.e. find the point $(w, 10)$ which someone starting at $(x, 10)$ should take to get to the mall in the shortest amount of time. There is usually some discussion about w being different for different values of x, so we encourage students to experiment with different numbers for w with follow-up analysis. Some students go right for the general approach with an arbitrary w. Using 'Distance = Rate × Time' they construct a function (of w) for the time T it takes to go from $(x, 10)$ through $(w, 10)$ to the mall:

$$T(w) = \frac{x - w}{55} + \frac{\sqrt{10^2 + w^2}}{30}$$

Performing the usual optimization task of taking the derivative with respect to w and setting this equal to 0 yields

$$T'(w) = \frac{-1}{55} + \frac{w}{30\sqrt{100 + w^2}} = 0$$

The students find that the optimal value of w is 6.50701 and that this position is independent of x, indeed, note that x does not appear in the derivative, $T'(w)$. Thus, the minimal time it would take to go from the point of intersection (call it $(z, 10)$) of the circular boundary and the highway is $T(6.50701) = 0.79364$ hours, using $x = z = \sqrt{30^2 - 10^2} = 28.2843$ in our formulation of $T(w)$ above. This does not leave much time for highway driving beyond the 30-mile circle! Of course, one can play with the speeds in your own modified problem construction to make different geometric results here and elsewhere in the problem. Now knowing that we have $1 - 0.79364 = 0.20636$ hours left to travel on the highway we see that we can travel some $0.20636 \times 55 = 11.3498$ miles beyond the point $(z, 10) = (28.2843, 10)$. Thus we can travel all the way out to the point $(28.2843 + 11.3498, 10) = (39.6341, 10)$, the farthest point we can go east along the highway and still make it to the mall in an hour, in fact, exactly an hour. Call this point U. See figure 2.

We now know what is the best cut from the highway to the mall and how far out along the highway we can go. It is time to determine the boundary of those points outside the 30 mile radius circle surrounding the mall and off the highway from which we can get to the mall in an hour or less.

We first need to determine just how we would approach the highway from some arbitrary point outside the 30 mile circle and off the highway. For this we refer to figure 3. One way to formally do this is to construct a function from point $Q = (a, b)$, say, off the highway which is beyond the circle and northeast of the mall to see what is the best cut off ('cut-on') angle to enter the highway, i.e. what point $R = (c, 10)$ on the highway will assure minimum time from Q to R, thence along the highway to $P = (w, 10) = (6.50701, 0)$ and finally down to the mall at $(0, 0)$. If one does this using optimization techniques to minimize the time as above, one can prove that the path from Q to R is parallel to the path from P to $(0, 0)$ (the mall). That is, the angle at which one approaches the highway is the same as the one which one uses to leave the highway or in terms of lines, the roads OP and RQ are parallel. Often, students will want to list this fact in their assumptions because they cannot verify it, but they 'know it.' Some will take several specific points and 'prove' the fact to themselves. In class we enter a lively discussion about what one knows and what one believes in this setting as well as how one proves things to intellectual self-satisfaction.

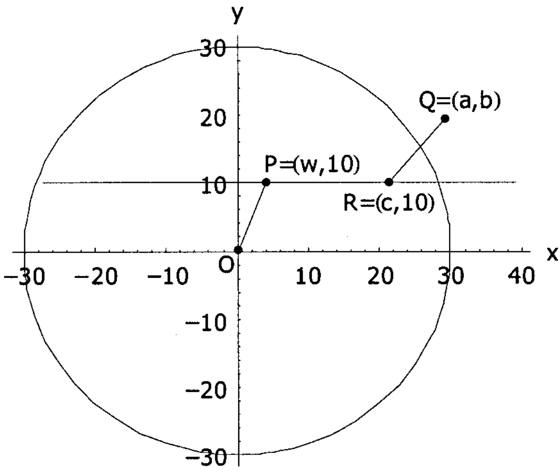

Figure 3. Mall figure with $O =$ Mall, $P =$ entry point to highway, $Q =$ typical outlying point, $R =$ entry point to highway.

7.1. *Data collection approach to finding the boundary*

In order to determine the boundary of those points outside the 30-mile radius circle surrounding the mall from which we can get to the mall in an hour or less, students often resort to sophisticated trial-and-error techniques. Indeed, they will try out some points beyond the circle northeast of the mall to see how long it takes to get to the mall using the 'parallel' approach to the highway discussed above. They will narrow this to points that are exactly one hour from the mall. Some students will take several such points and use a spreadsheet to fit a function (linear!) to this data. Indeed, they will often include the extremal point $(39.6341, 10)$ for it is on this boundary as well. Other students will presume this boundary is a line, because they want it to be a line. They will then find one point by trial and error and connect it to $(39.6341, 10)$ to form the line whose equation they can obtain through point-slope methods. Some students will just connect the point $(39.6341, 10)$ to the point of intersection of the optimal road (OP extended) away from the highway and the 30-mile radius circle. Some students find two points by trial and error, neither of which is $(39.6341, 10)$, to form the boundary line. There are a great many strategies offered, and a discussion of relative and absolute merit usually ensues.

7.2. *Analytical approach to finding the boundary*

A more sophisticated approach is to pick a general point, say $(w + d, 10 + a)$, that is d miles past the cutoff point on the highway, $(w, 10)$, and a miles off (north) of the highway. See figure 4. Write out the function

$$\text{time}(x) = \frac{\sqrt{w^2 + 10^2}}{30} + \frac{x}{55} + \frac{\sqrt{a^2 + (d - x)^2}}{30}$$

for the time it takes to get to the mall from the point $(w + d, 10 + a)$ in terms of some point $(w + x, 10)$ at which we enter the highway. Then minimize this in terms of the variable x. Indeed, one obtains x as a function of a and d, i.e. $x = -0.650791a + d$. Substituting this back into time(x) and then setting

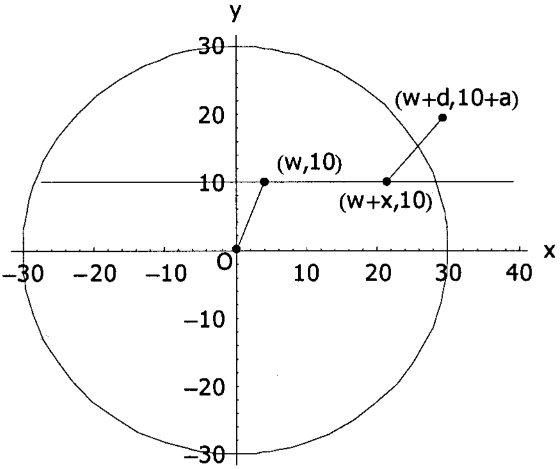

Figure 4. Mall figure for locating the point $(w + x, 10)$ at which one should enter the highway in order to minimize the time to get from point $(w + d, 10 + a)$ to the mall.

time(x) $= 1$, one can eliminate the variable x and obtain a relationship between a and d which must be satisfied for those points $(w + d, 10 + a)$ which are exactly one hour away from the mall. This linear function is $a = 21.5582 - 0.650791d$. From this one can obtain a relationship between the general x–y coordinates used in setting up the mall to obtain a true boundary equation $y = 35.7935 - 0.650791x$. It is a routine matter to reflect this line about the highway line $y = 10$ to obtain a boundary south of the highway.

7.3. *Final passage to solution*

Once we know the actual equations of the boundary lines, we can determine the points of intersection of these lines and the original boundary of the neighbourhood of the mall, i.e. the circle of radius 30 miles centred at $(0, 0)$. Using these points and the bounding equations, i.e. upper line, circle, and lower line (see figure 5), we develop the necessary integrals to obtain the additional area to add to the circular area (on the east side of the mall) and then double this to obtain the total additional area of 201.871 square miles. This, when added to the original area of $\pi(30)^2 = 2827.43$ square miles, gives a new mall neighbourhood of 3029.40 square miles for a modest 7.14% increase.

We offer up a sketch (see figure 5) of our area of interest, i.e. the eastern side of the mall, and note there is a symmetric western side of the mall area.

8. Extensions

This problem about the mall is a challenging one, but we might wish to add something to make it even more interesting. At least after solving the original problem it might be worth considering some of the following ‘complications’ which could occur. Students would have invested a great deal in the problem by the time these twists are suggested and hence would appreciate the higher levels of complexity, even if not carried out. Consider the following extensions of the problem:

(1) Ask for the additional area as a function of the speed permitted on the

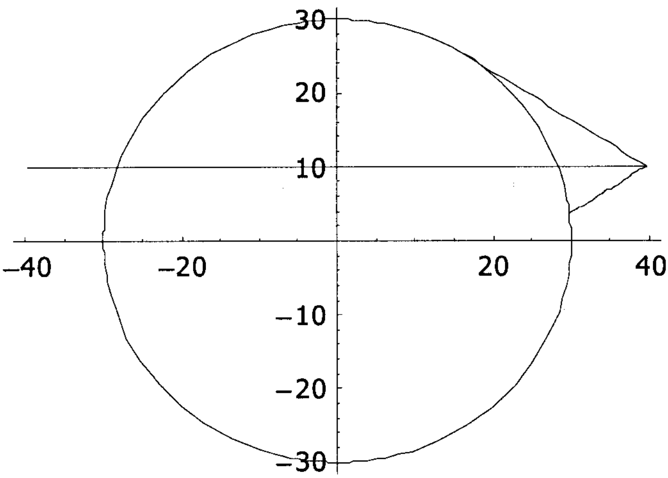

Figure 5. A sketch of the additional 'triangular' or 'wing' shape mall neighbourhood area added to the original circular area in the northeast. There is a comparable, but symmetric area in the northwest as well.

highway. So if we increase the speed on the highway what change occurs in the area of the resulting neighbourhood of the mall?

(2) Find the other new 'isotime' boundaries, e.g. what do all the points for which you can get to the mall from them in exactly 200 minutes look like in this new situation; in 120 minutes; in 75 minutes, etc.?
(3) Consider the original problem with a more realistic square grid of roads, i.e. do not allow mall bound traffic to be 'as the crow flies,' but rather restrict traffic to a reasonable road grid of avenues and streets. The students must first find out how to determine distances and times between points in this new grid.
(4) As an additional alternative to this square grid of roads, suppose the highway does not run parallel to the roads in the existing road grid.
(5) Consider differently-shaped highways, e.g. a section of a circular road with specific radius but not centred at the mall, perhaps centred on downtown. This could be a beltway around the city. Indeed, upon completion of the project the author placed a sinusoidal highway on the mall and the groans were quite audible, indicating they understood the increased complexity of this new problem.
(6) Introduce a second highway with a different speed.
(7) Identify the points inside the circular neighbourhood which would use the highway to reduce time. Discuss the changes in driving patterns that would result inside the circular neighbourhood.
(8) Suppose the costs to move materials are different in various sections of the region and find all the 'equicost' lines for a given cost. This might be of interest in moving goods to sell into the mall from surrounding areas.

9. Conclusion

We have introduced a problem and a process to illustrate our belief that complex, technology-based problems are worthy of our students' time. We have demonstrated the great potential for investigation, for differing approaches, and for interaction among students. The particular problem we posed is a rich one for student investigation and there are many more like it to be found in the dialogue between teachers, in reading non-mathematics source material, and in modifying existing problem opportunities. We encourage the reader to try such problems with their students.

Acknowledgments

The author first got the idea for this project from a presentation by the late Professor Don Koehler, Department of Mathematics, Miami University, Oxford OH, over 25 years ago. Professor Koehler based his talk on material found in *Geographic Studies of Urban Transportation*, which appears in the series, Studies in Geography, Number 16, 1968. Frank Horton, Editor, Northwestern University, pp. 128-170 [3].

The author is indebted to all those students who have tackled this and other problems of a complex, technology-based type for their ingenious strategies, their out-of-the-box thinking which led to a new view of a situation, and for their many 'Aha!' responses which make creating such scenarios worthwhile.

Finally, the author is thankful for the constructive reads of an early draft of this paper by Aaron Klebanoff, Don Mills, Bill Wilhelm and Doan Winkel.

Resources

A *Mathematica* notebook and an ASCII version of a problem similar to this problem, with solution and comments is available (under the title 'Malled'), as a part of a larger National Science Foundation project effort, 'Development Site for Complex, Technology-Based Problems in Calculus,' NSF Grant DUE-9352849. This notebook, along with other problem source material developed with support of the grant, is available on the World Wide Web: http://www.rose-hulman.edu/Class/CalculusProbs. Web site preparation was done by Dr Aaron Klebanoff, a bright, young colleague at Rose-Hulman Institute of Technology.

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